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TOWARDS A PROOF OF THURSTON'S GEOMETRIZATION THEOREM FOR ORBIFOLDS

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§ 0 Preface

This is a very informal note based on our discussion concerning Thurston's geometrization theorem for orbifolds mainly at Topology seminar of Tokyo Metropolitan University in 1984/85. The main purpose of that seminar was to understand its proof. The discussion at the seminar was based on the arguments Thurston gave in his course of 1984 spring, which the first author had attended. There, Thurston had described a basic idea for the proof and some details.

Our intention was thus to fill up logically and reasonablly understandable details. However, we have quite often faced difficulties of translating his idea to "usual" mathematics for us. The main cause seems the lack of terminologies to describe it. Consequently we could not complete our original intention.

The purpose of this note is thus to describe only our understanging of Thurston's idea until the deadline of submission of this article. We have tried to give a careful explanation as much as we can except §8. The arguments in the first five sections are faithfully based on Thurston's lecture. The arguments in §6 and §7 are rather selfish interpretation of his assertion. For example, Proposition 6.1 is our translation of Thurston's word, "one rescalling factor will work everywhere". We gave up to complete the last section, since there are obviously crucial assertions of which we have not been able to understand the proof. We just reminded his basic idea there.

Since our discussion was held without Thurston, any mistakes are ovbiously due to us. We are very willing to hear any suggestions, comments, criticism, pointing out mistakes .....

§ 1 Background and Theorem

We first recall the basic concepts of the orbifold and then state the geometrization theorem for orbifolds.

Definition : The orbifold O is a topological space with a structure locally modeled on $\mathbb{R}^n$ modulo a finite group action. More precisely O is an underlying topological space X_O with a system of local charts $\{(U_i, \varphi_i)\}$ in the sense that

(1) $\{U_i\}$ is an open cover of X_O closed under the intersection,

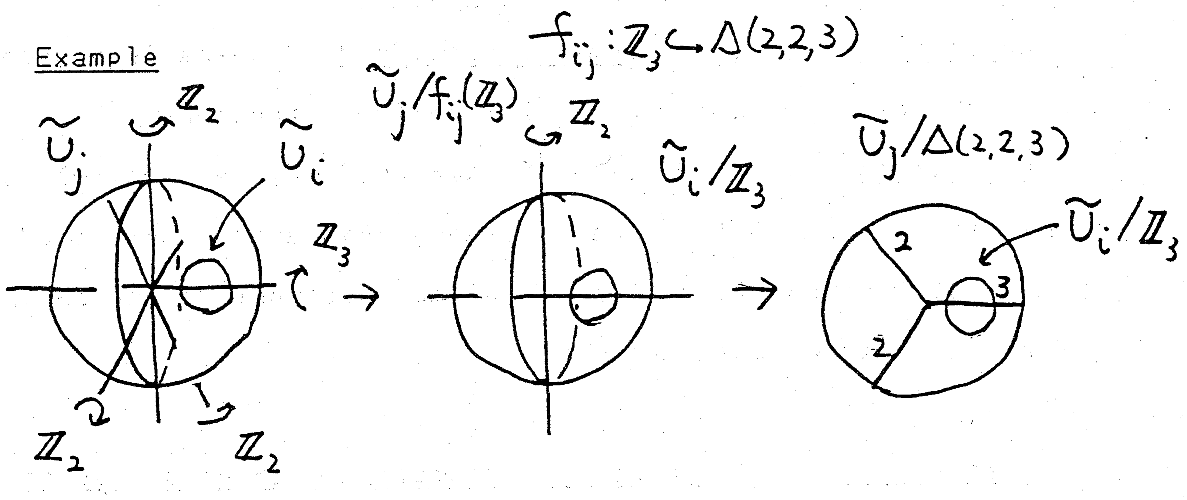
(2) for each chart (U_i, φ_i), there are an open set $\tilde{U}_i$ in $\mathbb{R}^n$ and a finite group Γ_i faithfully and diffeomorphically acting on $\tilde{U}_i$ so that $\varphi_i : U_i \to \tilde{U}_i/\Gamma_i$ is a homeomorphism, and

(3) if $U_i \subset U_j$, there are a monomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and an into homeomorphism $\varphi_{ij} : \tilde{U}_i \to \tilde{U}_j$ which make the following diagram commute,

$$\begin{tikzcd}
\varphi_{ij} : \tilde{U}_i \arrow[r] \arrow[d, "/\Gamma_i"] & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[r] & \tilde{U}_j/f_{ij}(\Gamma_i) \arrow[d] \\
 & \tilde{U}_j/\Gamma_j \\
U_i \arrow[r] \arrow[uu, "\varphi_i"] & U_j \arrow[u]
\end{tikzcd}$$

The set $\Sigma_O = \{ x \in X_O \mid$ the isotropy subgroup of $x \neq \{1\}\}$ is called a <u>singular</u> <u>locus</u>.

<u>Example</u>



<u>Definition</u> : The orbifold $\tilde{O}$ is a <u>covering</u> of an orbifold O if there is a projection $p : X_{\tilde{O}} \to X_O$ such that for any $x \in X_O$, there exists a neighborhood of x, $U = \tilde{U}/\Gamma$, so that for each component V_i of $p^{-1}(U)$, there is $\Gamma_i \subset \Gamma$ satisfying $V_i = \tilde{U}/\Gamma_i$.

<u>Definition</u> : The orbifold O is <u>good</u> if there is a covering of O without singular locus. It is <u>bad</u> otherwise.

<u>Definition</u> : Let (G, X) be a pair of a real analytic manifold X and a group of analytic diffeomorphisms G. The orbifold O is a (G, X)-<u>orbifold</u> if its charts are locally modeled on (G, X).

<u>Proposition 1.1</u> : <u>A</u> (G, X)-<u>orbifold</u> <u>is</u> <u>good</u>.

<u>Proof</u>. See [T] §13.

Assume that X is a smooth simply connected manifold and G is a group of diffeomorphisms which acts transitively on X with a compact isotropy subgroup for each $x \in X$. Then X admits a complete G-invariant riemannian metric. By Montgomery-Zippen's theorem [MG], G becomes a Lie group and $X = G/G_x$ becomes an analytic manifold with analytic action by G. Assuming further that G is maximal, we call (G, X) a *geometry*.

Definition : O is a *geometric orbifold* if there is a geometry (G, X) so that O admits a (G, X)-orbifold structure.

Definition : When O is a (G, X)-orbifold, let $\mathrm{vol}(O)$ be the volume of $X_O - \Sigma_O$ with respect to some G-invariant metric of X. Notice that the condition "finite volume" does not depend on the choice of a G-invariant metric.

Theorem 1.2 : *A 3-dimensional geometry which possesses at least one compact geometric orbifold is one of the Thurston's eight geometries.*

Proof. See [S].

Remark. For any 3-dimensional orbifold O, the valency of a vertex of Σ_O is 3.

Definition : $O \to O'$ is a *fibration* of orbifolds with generic fiber F if it is based on a continuous projection $f : X_O \to X_{O'}$, such that each point $x \in O'$ has a neighborhood $U = \tilde{U}/\Gamma$ (with $\tilde{U} \subset \mathbb{R}^n$) satisfying $f^{-1}(U) = \tilde{U} \times F/\Gamma$ for some action of Γ on F (where Γ acts by the diagonal action). The product structure is consistent with f : the diagram below must commute.

$$
\begin{tikzcd}
\tilde{U} \times F \arrow[r, "/\Gamma"] \arrow[d, "p_1"'] & f^{-1}(U) \arrow[d, "f"] \\
\tilde{U} \arrow[r, "/\Gamma"] & U .
\end{tikzcd}
$$

Theorem 1.3 : *A 3-dimensional orbifold which fibers over a 2-dimensional geometric orbifold of finite area belongs to Seifert* geometry of finite volume. A 3-dimensional orbifold which fibers over a compact 1-dimensional orbifold with euclidean fiber belongs to either euclidean, nilpotent or solvable geometry.

Proof. Apply the method in [S] or [K] to orbifolds.

This theorem will be used implicitly later without refer whenever we find a fibration. That is to say, when we find a fibration, it will mean automatically that the orbifold in question is geometric.

Definition : $O'^m \subset O^n$ is a suborbifold if it is a subspace $X_{O'} \subset X_O$ with an orbifold structure so that for each $x \in X_{O'}$, there is a neighborhood U_x in X_O such that $U_x = \tilde{U}/\Gamma_x$, $\tilde{U} \cap \mathbb{R}^m = \tilde{V}$ is Γ_x-invariant and $(U_x, U_x \cap X_{O'}) = (\tilde{U}, \tilde{V})/\Gamma_x$.

Definition : A 3-dimensional orbifold O is irreducible if every spherical 2-suborbifold of O bounds an orbifold of the form B^3/Γ where $\Gamma \subset SO(3)$.

Definition : A non-spherical 2-suborbifold O^2 of O^3 is incompressible if a simple closed curve of O^2 which bounds either a disk or a disk with an elliptic singular point in $O^3 - O^2$ bounds it also in O^2.

The following is Thurston's geometrization theorem for orbifolds of which we shall describe the (unfortunately incomplete) proof in the sequel.

The main theorem : Let O be a compact orientable irreducible 3-orbifold with (possibly empty) incompressible euclidean boundary. If O admits no bad 2-suborbifold and $\dim \Sigma_O = 1$, then there is a finite (possiblly empty) collection of mutually disjoint incompressible euclidean 2-suborbifolds $O_1, \ldots, O_n$ such that each component of $O - O_1 \cup \ldots \cup O_n$ is a geometric 3-orbifold. In particular, O is good.

This big theorem has a quite many corollaries, however we state only one which is towards Thurston's geometrization conjecture.

Corollary : If a closed orientable prime 3-manifold M admits a nontrivial orientation preserving periodic map with nonempty fixed point set, then M admits a geometric decomposition.

Proof. Let O be the quotient orbifold of M by the periodic map. Then by the theorem, O splits into geometric orbifolds. Pulling back these structures to M, we get a geometric decomposition as desired.

§ 2 The first reduction and hyperbolic 3-cone-manifolds

In what follows, we denote by O the orbifold that satisfies the condition of the main theorem. Let U_i be a small ball neighborhood of each vertex $v_i \in \Sigma_O$ and denote $X_O - \cup U_i$ - {toral boundary} by N and $N \cap \Sigma_O$ by Σ_N. The boundary of N consists of 2-spheres with three or four punctures. Our first reduction is,

The first reduction : If the theorem is true provided that $N - \Sigma_N$ admits a complete hyperbolic structure of finite volume with totally geodesic boundary, then the theorem is true in general.

To see this, we refer to the following uniformization theorem for Haken manifolds which is in our convenient form [Sm], [CS].

Theorem 2.1 : Let M be a Haken manifold without boundary but with incompressible toral end. Then M either contains a nonperipheral torus, is a Seifert fibered space or admits a complete hyperbolic structure of finite volume.

Proof of the first reduction : Let DN be the double of N. The complement $M = DN - D\Sigma_N$ is a Haken manifold without boundary but with incompressible toral end since O has no bad 2-suborbifold (teardrop in fact). Notice that M admits an obvious orientation reversing involution τ which reflects M along $F = \partial N - \Sigma_N$. F consists of 2-spheres with three or four punctures. By Theorem 2.1, we have three cases for M.

Case 1 : M contains an essential torus T_1. We first think of the case when $T_1 \subset N - \Sigma_N$. If T_1 is compressible in O. Then there is a compressing disk D with a cone point and $(T_1 - \mathcal{N}(D)) \cup D_1 \cup D_2$ becomes a spherical suborbifold O' where D_1 and D_2 are frontiers of $\mathcal{N}(D)$, Since O is irreducible, O' must bound $B^3/\mathbb{Z}_n$. This means that T_1 is either ∂-parallel or compressible, and we get contradiction. Thus T_1 must be incompressible in O. Then by cutting O along T_1, we get a new orbifold which still satisfies the condition of the main theorem. By Haken's finiteness property of incompressible surfaces for $N - \Sigma_N$, we have reduced hierarchy to get simple pieces, which is the last two cases in Theorem 2.1.

We next deal with the case when T_1 intersects F essentially and minimally. We then have two cases. One is that some intersection circle bounds a puncture in F. The other is that any intersection circle bounds two punctures in F. We first deal with the former case.

In this case, we can find a properly embedded annulus $A \subset N$ which is a part of T_1 and whose boundary has that circle as a component. The other component bounds one or two punctures. When it bounds two punctures, it is on a component of F which inherits an euclidean boundary of O, and also bounds a disk with a cone in O. This contradicts the incompressibility of ∂O. Hence we may assume that the other component of ∂A bounds a puncture on F. Now fill up ∂A by two disks with a cone in O, and get a spherical suborbifold $O' \subset O$ since O has no bad suborbifold. Then O' bounds $B^3/\mathbb{Z}_n$ in O since O is irreducible, and hence A becomes parallel to $\partial\mathcal{N}(\Sigma_N)$. Do the same argument to an adjacent essential annulus again and again and we finally conclude that T_1 is ∂-parallel. This is contradiction.

Let us deal with the case when any intersection circle bounds two punctures in F. In this case, any intersecting component of F must be a four punctured sphere which inherits an euclidean boundary of O. Since T_1 intersects essentially with F, we can again find a properly embedded annulus A in N which is a part of T_1. Fill up ∂A by two disks with two cones on ∂O and get an euclidean suborbifold $O' \subset O$. Suppose that O' is incompressible and not ∂-parallel. Then pushing O' into the interior of O and cutting O along it, we get a new orbifold which still satisfies the assumption of the theorem. Thus we have reduced to the simple cases similarly as before. If O' is compressible or ∂-parallel, then O' bounds a suborbifold P to which the fibration : $A = S^1 \times I \to I$ extends. We next fill up ∂A by the complementary region of the disks used previously and get another euclidean suborbifold O''

$\subset O$. O'' is a torus or $S^2(\pi,\pi,\pi,\pi)$ according to whether ∂A stays on a component of ∂O or not. If O'' is incompressible and not ∂-parallel, then we are done similarly as before. Thus the rest is the case when both O' and O'' are compressible or ∂-parallel. If O'' is compressible or ∂-parallel, then O'' bounds a suborbifold Q again to which the fibration : $A = S^1 \times I \to I$ extends. Moreover Q is the complement of P in O. This means that $O = P \cup Q$ admits a circle fibration as an orbifold. In particular, O is geometric.

<u>Case 2</u> : M admits a Seifert fibration. Suppose $F \neq \phi$. We may assume that τ preserves the fibration. Since F cannot be fibered, a fiber must intersect transversally with F. Since F is the fixed point set of τ, τ maps each fiber to itself. Thus $N - \Sigma_N$ has an I-bundle structure and hence (N, Σ_N) must be homeomorphic to $(S^2$, 3 or 4 pts) $\times I$. This shows that O admits a spherical or euclidean orbifold structure.

Assume $F = \phi$. Then M is disconnected and Σ_N forms a link in N. Subcase a) when each fiber on ∂M is not homotopic to meridians of the link. Then the fibration extends to a fibration on N and O becomes a fibered orbifold, and we are done. Subcace b) when some fiber on ∂M is homotopic to a meridian of a component of Σ_N. If a base orbifold of a component containing this fiber is neither D^2 nor D^2 with a cone, then by elementary cut and paste construction using meridian disks, we get an essential spherical suborbifold in O. However since O was irreducible, this is impossible. Therefore a base orbifold must be either D^2 or D^2 with a cone. Since it must be homeomorphic to $S^1 \times D^2$, by changing fibration, we can reduce this case to the subcase a).

<u>Case 3</u> : int M admits a complete hyperbolic structure of finite volume. Then τ is homotopic to an isometry τ_0 which fixes a surface homotopic to F. Since this surface is the fixed point set of an orientation reversing isometric involution, it is totally geodesic and we may conclude that $N - \Sigma_N$ admits a complete hyperbolic structure of finite volume with totally geodesic boundary F. This condition is the starting point of the whole argument in what follows.

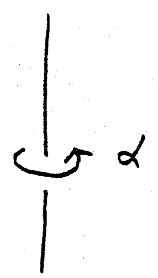
We have reduced the argument to the case when $N - \Sigma_N$ admits a complete hyperbolic struture of finite volume with totally geodesic boundary. We further assume for simplicity that O is closed, that is to say, N is compact and $\partial N - \Sigma_N$ consists of three punctured spheres. This is just for simplicity and will not be essential restriction. Let us emphasize our starting point under this condition again.

<u>Initial setting</u> : $N - \Sigma_N$ <u>admits a complete hyperbolic structure of finite volume with totaly geodesic boundary, where</u> N <u>is compact and</u> $\partial N - \Sigma_N$ <u>consists of three punctured spheres</u>.

The main idea is to deform a complete hyperbolic structure on $N - \Sigma_N$ to a geometric structure of O continuously along geometric (mainly hyperbolic) spaces only with cone type singularities. To say more precisely, we need explicitly to define the cone-manifold structure. The cone-manifold structure is a generalized concept of the orbifold structure. It is locally modelled on not necessarily $\mathbb{R}^n$ modulo a finite group action but on identification of a rotation along some axis with arbitrary angle and their combination. More visually for hyperbolic case, we have

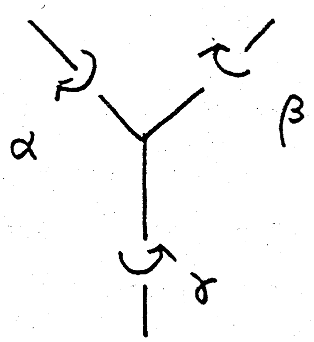
<u>Definition</u> : A <u>hyperbolic</u> 3-<u>cone</u>-<u>manifold</u> C is a topological space X_C with a singular hyperbolic structure locally modelled on one of the following sets.

(1) A neighborhood of an elliptic axis of cone angle α $(0<\alpha\leq 2\pi)$.

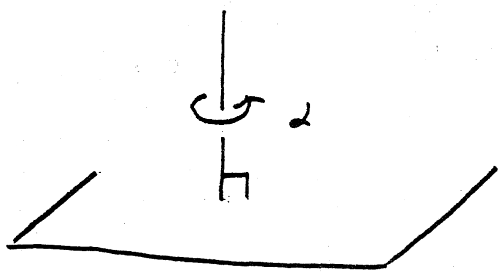


A neighborhood of an elliptic axis of cone angle = 2π corresponds to a neighborhood of a non singular point. When α tends to zero, the structure approaches a neighborhood of a cusp.

(2) A neighborhood of the vertex where more than two elliptic axis meet. The summation of their cone angles are assumed to be greater than $2(n-2)\pi$ where n is the number of axis which meet there.



(3) A half neighborhood of an elliptic axis splitted by a totally geodesic plane which intersects perpendicularly. This is a model for the points on boundary and in particular the boundary is a totally geodesic hyperbolic 2-cone-manifold.



The set $\Sigma_C = \{ x \in X_C \mid x$ is on some singular elliptic axis or a vertex$\}$ is called a singular locus.

Since we are only interested in a special type of hyperbolic 3-cone-manifolds, we just describe such for better understanding. What we are interested in is a hyperbolic 3-cone-manifold of finite volume with spherical totally geodesic boundary. Furthermore, the case when exactly three axis meet at each vertex, exactly three axis meet at nontoral infinity (end) and exactly three axis meet each component of boundary, is in our exclusive concern. Let α, β and γ be cone angles of three axis which are subject to the above situation. Then whether $\alpha+\beta+\gamma$ is $>$, $=$ or $<$ 2π reflects to whether those three axis meet at the vertex, at infinity or they meet the common boundary component. Although we get at the limit of deformation a cone-manifold with an end where four axis meet, we always assume that our hyperbolic 3-cone-manifold is of this type without specifying otherwise. We note that since our cone-manifold is of finite volume, its end is either a usual toral one or

$S^2(\alpha,\beta,\gamma) \times [0,\infty)$ where $\alpha+\beta+\gamma = 2\pi$. The boundary must be a hyperbolic 2-cone-manifold based on S^2 with three cone points.

We leave the precise definition of another geometric cone-manifold to the reader since what we need is fairly simple rather than its complicated strict definition.

Definition : The holonomy of C is a holonomy of a hyperbolic structure on $X_C - \Sigma_C$. That is a representation $\rho : \pi_1(X_C - \Sigma_C) \to PSL_2\mathbb{C}$.

Definition : A combinatorial type Δ_C of a hyperbolic 3-cone-manifold C is a pair of topological spaces obtained from (X_C, Σ_C) by collapsing boundaries and compactifying each nontoral ends to one point.

Remark : If cone angles are large enough, then (X_C, Σ_C) and its combinatorial type are topologically same.

Exceptional definition : We shall define the combinatorial type of a complete hyperbolic manifold based on $N - \Sigma_N$ in the first reduction to be (X_0, Σ_0) just for our convenience.

We first wish to see that any structure can be slightly changed along small algebraic deformation. To see this, we first need lemmas in hyperbolic geometry.

Lemma 2.2 : Let α and β be elliptic elements of Isom $\mathbb{H}^3$ with axis ℓ_α and ℓ_β. Then if $\beta\alpha$ is elliptic, then there is a totally geodesic plane P in $\mathbb{H}^3$ which contains $\ell_\alpha \cup \ell_\beta$.

Proof. Assume the contrary and choose the shortest geodesic segment $s = \overline{xy}$ which connects ℓ_α and ℓ_β. Let R be a plane containing s and perpendicularly intersect with ℓ_α at x, and let Q be a plane containing s and perpendicularly intersect with

ℓ_β at y. Let ν_1 and ν_2 be geodesics on R which intersect s with angle = rotation angle of $\pm\alpha^{1/2}$ at x and let λ_1 and λ_2 be geodesics on Q which intersect s with angle = rotation angle of $\pm\beta^{1/2}$ at y. Again let $t_1 = \overline{a_1 b_1}$ and $t_2 = \overline{a_2 b_2}$ be the shortest geodesic segments which connect ν_1 and λ_1, ν_2 and λ_2 respectively. Then if we let r be a 180 degree rotation around s, then $r(t_1) = t_2$ and also we have $d(x,a_1) = d(r(x),r(a_1)) = d(x,a_2)$. Therefore $\alpha(a_1) = a_2$. Similarly $\beta(b_2) = b_1$. Since $r(t_1) = t_2$ and $r(Q) = Q$, the angle between t_1 and Q = the angle between t_2 and Q. Hence t_1 and βt_2 are on the same geodesic. Similarly $\alpha^{-1}(t_2)$, t_1 and hence $\beta(t_2)$ are on the same geodesic ℓ. Therefore since $\beta\alpha(\ell) \cap \ell \supset \alpha^{-1}(t_2)$, $\beta\alpha(\ell) = \ell$. This means that $\beta\alpha$ is a loxodromic transformation, which is contradiction.

<u>Lemma 2.3</u> : <u>Let</u> α <u>and</u> β <u>be parabolics with</u> $\mathrm{fix}(\alpha) \neq \mathrm{fix}(\beta)$ <u>and suppose</u> $\alpha\beta$ <u>is a parabolic</u>. <u>Then there is a totally geodesic plane</u> P <u>with</u> $\alpha(P) = \beta(P) = P$.

<u>Proof</u>. By changing basis, we may assume that $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$. Then $\alpha\beta = \begin{pmatrix} 1+b & 2 \\ b & 1 \end{pmatrix}$ and $\mathrm{trace}(\alpha\beta) = 2 + b$. Since $\alpha\beta$ is parabolic, $b = -4$. Thus we can easily see that the plane on the real line is invariant.

We now discuss the existence of geometric deformation in general. Let (X, Σ) be an underlying space of some hyperbolic 3-cone-manifold, let $e_1, \dots, e_k$ be edges of Σ and let $c_1, \dots, c_m$ be circle components of Σ. Denote by $\mu_1, \dots, \mu_{k+m}$ the elements of $\pi_1(X - \Sigma)$ which correspond to meridians of $e_1, \dots,$

e_k, c_1, ..., c_m respectively.

Theorem 2.4 : Let $\mathcal{S}_0$ be a hyperbolic cone-manifold structure on (X, Σ) or a complete hyperbolic structure on $X - \Sigma$ and let ρ_0 be its holonomy representation. If ρ is sufficiently close to ρ_0 in $R(\pi_1(X - \Sigma))$ with $\rho(\mu_i)$ elliptic (or parabolic), then there is a hyperbolic cone-manifold with the same combinatorial type such that

(1) its structure $\mathcal{S}_1$ is close to $\mathcal{S}_0$ and

(2) its holonomy representation is ρ.

Proof. Denote $X - \mathcal{N}(\partial X \cup \Sigma)$ by U. Then by [T], there is a hyperbolic structure $\mathcal{S}_1$ on U which is close to $\mathcal{S}_0|U$ and whose holonomy is ρ. We will first show that if $\mathcal{S}_0$ is a cone-manifold structure rather than a complete structure, $\mathcal{S}_1$ actually extends to a hyperbolic cone-manifold structure on the topological space (X, Σ) when the structure $\mathcal{S}_0$ has no cusp end. Then we will see that when $\mathcal{S}_0$ has a cusp end, the structure $\mathcal{S}_1$ extends to a cone-manifold structure on slightly modified space according to whether a cusp comes into interior or blows up at infinity. Lastly we modify those proof to cover when $\mathcal{S}_0$ is a complete hyperbolic structure.

Since $\rho(\mu_i)$ is elliptic, $\mathcal{S}_1$ extends to all of X except on neighborhoods of vertices of Σ and ∂X. If the three edges e_1, e_2, e_3 intersect at v in the structure $\mathcal{S}_0$, then since μ_1, μ_2, are elliptic and the product $\mu_1\mu_2 = \mu_3$ is also elliptic, there is a totally geodesic plane $P \supset \ell_{\mu_1} \cup \ell_{\mu_2}$ under the structure $\mathcal{S}_1$ by Lemma 2.2. We may also assume that ℓ_{μ_1} intersects with ℓ_{μ_2} because $\mathcal{S}_1$ is sufficiently close to $\mathcal{S}_0$.

Thus three axis ℓ_{μ_1}, ℓ_{μ_2} and ℓ_{μ_3} meet at one point. This means that $\mathcal{S}_1$ extends to a neighborhood of v. If e_1, e_2 and e_3 do not intersect even at infinity but μ_3 is the product $\mu_1\mu_2$, then there are totally geodesic planes P_1, P_2 and P_3 so that $P_i \cap P_j = e_k$, $\{i,j,k\} = \{1,2,3\}$ by Lemma 2.2. Again we may assume that $P_1 \cap P_2 \cap P_3 = \phi$ since $\mathcal{S}_1$ is sufficiently close to $\mathcal{S}_0$. Then by easy hyperbolic geometry, there is a unique totally geodesic plane P which intersects perpendicularly to all of P_i's. This means that $\mathcal{S}_1$ extends up to ∂X where P comes down as a totally geodesic boundary. When e_1, e_2 and e_3 intersect at infinity, they form a cuspidal end. However mixing those arguments according to how cone angles are changing, we can see the claim. When $\mathcal{S}_0$ is a complete hyperbolic structure, we follow the last argument using Lemma 2.3 instead of Lemma 2.2.

Let us go back to our situation. We had a topological space (N, Σ_N). We first name their edges and circle components of Σ_N in the same way as Σ above. Our starting point was that $N - \Sigma_N$ admits a complete hyperbolic structure. Let ρ_0 be the holonomy of this structure, then $\rho_0(e_i)$'s and $\rho_0(c_j)$'s all are parabolic. By Theorem 2.4, we have a small deformation of hyperbolic structure keeping its combinatorial type constant so that a cone angle of each axis increases if there is such an algebraic deformation. Thus we may expect to have deformation ρ_t to the structure so that the cone angles around meridians are equal to $2\pi/n_i$ where n_i is the order of the isotropy subgroup of the i-th axis of Σ_0, provided there is an algebraic deformation. It is nothing but a hyperbolic orbifold structure on O. This is impossible in general however this tells us the target of the deformation we construct. We thus wish to see how we can deform algebraically next.

Let us denote by Π the fundamental group $\pi_1(N - \Sigma_N) = \pi_1(X_0 - \Sigma_0)$. Then ρ_0 can be lifted to a representation to $SL_2\mathbb{C}$ by the argument in [CS]. Let $R(\Pi)$ be a complex affine variety formed by all the representation of Π to $SL_2\mathbb{C}$. Then there is a basic fact,

<u>Theorem 2.5</u> : <u>Let</u> R_0 <u>be an irreducible component of</u> $R(\Pi)$ <u>containing</u> ρ_0. <u>Then</u>

$$\dim_{\mathbb{C}} R_0 \geqq -3\chi(\Sigma) + m + 3$$

<u>where</u> m <u>is the number of circle components of</u> Σ.

<u>Proof</u>. See [T] or [CS].

Let μ_i be again a meridian of the i-th axis of Σ_0. Then since μ_i is assumed to be elliptic during a small deformation ρ_t of Theorem 2.4, trace $\rho_t(\mu_i) = 2\cos\theta/2$ where θ is a cone angle at t. Thus the final target should have the value $2\cos\pi/n_i$. Let $f : R_0 \to \mathbb{C}^{k+m}$ be a regular map defined by $f(\rho) = (\mathrm{trace}(\mu_1), \ldots, \mathrm{trace}(\mu_{k+m}))$. Assuming that $\rho_0(\mu_i)$ all are lift to $SL_2\mathbb{C}$ with trace = 2, we will find an algebraic path of deformation starting from such a representation. Let $L \subset \mathbb{C}^{k+m}$ be a complex line containing $p = (2, \ldots, 2)$ and $q = (2\cos\pi/n_1, \ldots, 2\cos\pi/n_{k+m})$ and let $\ell(t)$ be a real line in L parametrized by t such that $\ell(0) = p$ and $\ell(1) = q$.

<u>Theorem 2.6</u> : <u>There are a complex algebraic curve</u> D <u>in</u> R_0 <u>and a piecewise real algebraic curve</u> $\gamma : [0, \bar{t}) \to D$ $(0 < \bar{t} \leqq \infty)$

<u>such that</u>

(1) $\gamma(0)$ <u>is a holonomy of a complete hyperbolic structure,</u>

(2) $f(\gamma(t)) = \ell(t)$ <u>and</u>

(3) γ <u>is maximal under</u> (1) <u>and</u> (2).

<u>Proof</u>. Since #(edges of Σ_N) = k, #(vertices of Σ_N) = 2k/3 and hence $\chi(\Sigma_N) = -k/3$. Since $\dim_{\mathbb{C}} R_0 \geqq -3\chi(\Sigma_N)+m+3$, we have $\dim_{\mathbb{C}} R_0 \geqq k+m+3$. Now L is of dimension one, and hence each component of $f^{-1}(L)$ has dimension $\geqq \dim_{\mathbb{C}} R_0-(k+m)+1 \geqq 4$. Take a component C containing ρ_0. By Lok's local rigidity [L], a neighborhood of ρ_0 in $f^{-1}(p)$ has complex dimension 3. We can choose a complex curve D in C which contains ρ_0 and of which the image by f is Zariski dense in L. Then we are almost done since $f^{-1}(\ell) \cap D$ form piecewise real algebraic curves with which we can choose a required path.

By Theorem 2.4 and 2.6, we eventually get a deformation ρ_t of hyperbolic structure along the path in the Theorem 2.6. The deformation starts at the complete structure and stays in cone-manifold structures keeping its combinatorial type constant. Let t_0 ($\leqq \bar{t}$) be the terminal time where we cannot go over by geometric deformation. That is the end of our deformation. If $t_0 > 1$, it means that we get a hyperbolic orbifold structure on O. The rest of our task is to see what happens otherwise. Thus we will always assume in the sequel that $t_0 \leqq 1$.

§ 3. <u>Limits of metric spaces</u>

In this section, we quickly review Gromov's theory which will be used to analyze the limit of our deformation.

<u>Definition</u> : Let X and Y be compact metric spaces. A relation $R \subset X \times Y$ is an ε-approximation between X and Y if

(1) $pr_X(R) = X$ and $pr_Y(R) = Y$ and

(2) For any $x, x' \in X$ and $y, y' \in Y$ with xRy and $x'Ry'$, we have $|d(x, x') - d(y, y')| < \varepsilon$.

<u>Definition</u> : Let X_i $(i = 1, 2, \ldots)$ and Y be metric spaces. X_i converges to Y $(X_i \to Y)$ if for any ε, there is i_0 such that if $i > i_0$, then there is an ε-approximation $R_{i,\varepsilon}$ between X_i and Y.

<u>Proposition 3.1</u> : <u>Let</u> X <u>and</u> Y <u>be compact metric spaces</u>. <u>If there is an</u> ε-<u>approximation between</u> X <u>and</u> Y <u>for any</u> ε, <u>then</u> X <u>is isometric to</u> Y.

<u>Proof</u>. Choose $\{x_i\} \subset X$ so that for any $n \in \mathbb{N}$, the subset of the first k_n points, $\{x_1, \ldots, x_{k_n}\}$, forms a $1/n$-net, i.e., $\cup_{i=1}^{k_n} B_{1/n}(x_i, X) = X$. Take $y_i^{(n)} \in Y$ so that $x_i R_{1/n} y_i^{(n)}$. Since Y is compact, there is a subsequence $\{n\} \supset \{n_1\}$ such that $y_1^{(n_1)}$ converges. Define $I(x_1)$ to be $\lim y_1^{(n_1)}$. Similarly take a subsequence $\{n_1\} \supset \{n_2\}$ such that $y_2^{(n_2)}$ converges and define $I(x_2)$ to be $\lim y_2^{(n_2)}$. This process eventually defines $I|\{x_i\}$.

We claim that $I|\{x_i\}$ is an isometry. To see this, fix $j, k \in \mathbb{N}$ so that $j < k$. Then $\{n_j\} \supset \{n_k\}$ and $I(x_j) = \lim y_j^{(n_j)} = \lim y_j^{(n_k)}$ and $I(x_k) = \lim y_k^{(n_k)}$. For any $n \in \{n_k\}$, we have relations $x_k R_{1/n} y_k^{(n)}$ and $x_j R_{1/n} y_j^{(n)}$ and hence $|d(x_k, x_j) - d(y_k^{(n)}, y_j^{(n)})| < 1/n$. Since n can be arbitrarily large in $\{n_k\}$, we have that $d(x_k, x_j) = d(y_k, y_j)$.

Obviously I is continuous. Since $\{x_i\}$ is a dense in X, we get an isometry of X to Y by extending I continuously.

Corollary 3.2 : *If* X_i *converges to compact metric spaces* Y_1 *and* Y_2, *then* Y_1 *is isometric to* Y_2.

Definition : Let (X, x_0) and (Y, y_0) be complete metric spaces with base points. A relation $R \subset X \times Y$ is an ε-approximation between (X, x_0) and (Y, y_0) if

(1) there is $y \in Y$ such that $x_0 R y$ and $d(y_0, y) < \varepsilon$,

(2) $pr_X(R) \supset B_{1/\varepsilon}(x_0, X)$, $pr_Y(R) \supset B_{1/\varepsilon}(y_0, Y)$ and

(3) $R \cap (B_{1/\varepsilon}(x_0, X) \times B_{1/\varepsilon}(y_0, Y))$ is an ε-approximation between $B_{1/\varepsilon}(x_0, X)$ and $B_{1/\varepsilon}(y_0, Y)$.

Definition : (X_i, x_i) converges to (Y, y) if for any ε, there is i_0 such that there is an ε-approximation between (X_i, x_i) and (Y, y) for all $i > i_0$.

Proposition 3.3 : *Let* X_i *and* Y *be complete metric spaces so that* $B_r(x_i, X_i)$ *is compact for each* i. *If* (X_i, x_i) *converges to* (Y, y), *then* $B_r(y, Y)$ *is compact*.

Proof. It is enough to show that $B_r(y, Y)$ is totally bounded. Let ε be an arbitrarily small number. Then since (X_i, x_i) converges to (Y, y), there is an $\varepsilon/2$-approximation between $B_r(x_k, X_k)$ and $B_r(y, Y)$ for some k. For each $x \in B_r(x_k, X_k)$, take $y(x)$ so that $xRy(x)$. Then $R(B_{\varepsilon/2}(x, B_r)) \subset B_\varepsilon(y(x), B_r)$. Hence if we let $\{z_1, .., z_m\}$ be a $\varepsilon/2$-net of $B_r(x_k, X_k)$, then $B_r(y, Y)$ is covered by $\bigcup_{j=1}^m B_\varepsilon(y(z_j), B_r(y))$, which shows nothing but $\{y(z_j)\}$ is an ε-net of $B_r(y, Y)$.

<u>Corollary 3.4</u> : <u>Let</u> (X_i, x_i), (Y_1, y_1) <u>and</u> (Y_2, y_2) <u>be compact metric spaces with base points</u>. <u>Suppose that</u> $B_r(x_i, X_i)$ <u>is compact for any</u> $r > 0$ <u>and</u> $i = 1, 2, \ldots,$ <u>and if</u> (X_i, y_i) <u>converges to</u> (Y_1, y_1) <u>and</u> (Y_2, y_2), <u>then</u> (Y_1, y_1) <u>is isometric to</u> (Y_2, y_2).

<u>Proof</u>. $B_r(y_1, Y_1)$ and $B_r(y_2, Y_2)$ are compact by Proposition 3.2. Thus by the previous corollary $B_r(y_1, Y_1)$ and $B_r(y_2, Y_2)$ are isometric. Since r was arbitrary, (Y_1, y_1) and (Y_2, y_2) turn out to be isometric.

<u>Theorem 3.5</u> (Gromov [G]) : <u>Let</u> $\{(X_i, x_i)\}$ <u>be a sequence of complete metric spaces so that</u> $B_R(x_i, X_i)$ <u>is compact for all</u> $R > 0$ <u>and</u> i. <u>Then the followings are equivalent</u>.

(1) <u>There is a subsequence</u> $\{j\} \subset \{i\}$ <u>so that</u> (X_j, x_j) <u>converges to a complete metric space</u> (Y, y).

(2) <u>There is a subsequence</u> $\{k\} \subset \{i\}$ <u>so that for any</u> $R > 0$ <u>and</u> $\varepsilon > 0$, <u>there is a constant</u> $K_{R,\varepsilon}$ <u>satisfying</u>

$$N(\varepsilon, B_R(x_k, X_k)) = \min \#\{\varepsilon\text{-balls covering } B_R(x_k, X_k)\} < K_{R,\varepsilon},$$

<u>where</u> $K_{R,\varepsilon}$ <u>depends only on</u> R <u>and</u> ε.

<u>Proof</u>. That (1) implies (2) is easy. Let $\{k\} = \{j\}$. By Proposition 3.2, $B_R(y, Y)$ is compact for any $R > 0$. If k is sufficiently large, there is an $\varepsilon/2$-approximation between $B_R(x_k, X_k)$ and $B_R(y, Y)$. Then $N(\varepsilon/2, B_R(y, Y)) \geq N(\varepsilon, B_R(x_k, X_k))$ and we are done.

Let us show that (2) implies (1). We may assume without loss of generality that $\{k\} = \{i\}$ and X_i is compact with $\operatorname{diam}(X_i) < 2R$ by looking only at $B_R(x_i, X_i)$. Let $\varepsilon_n = 2^{-n}$ and $K_n = K_{\varepsilon_n, R}$ which should be an integer. Let A_n be a finite set $\{a_{m_1 m_2 \cdots m_n} \mid$

m_p is an integer with $0 \leqq m_p \leqq K_p-1$ where $p = 1, 2, .., n$ }. Obviously $\# A_n = K_1 \times .. \times K_n$. There is a canonical embedding $A_n \subset A_{n+1}$ by identifying $a_{m_1 m_2 .. m_n}$ with $a_{m_1 m_2 .. m_n 0}$ and hence we have an increasing sequence $A_1 \subset A_2 \subset \ldots$ Now let $A = \overset{\infty}{\cup} A_n$.

Define a map $I_i : A \to X_i$ inductively so that $I_i(A_n)$ form an ε_n-net of X_i satisfying $I_i(a_{m_1 .. m_n}) \subset B_{\varepsilon_{n-1}}(a_{m_1 .. m_{n-1}})$. Let F be a set of real valued bounded functions on A with sup norm $\| \ \|$, and define a map $h_i : X_i \to F$ by $h_i(x)(a) = d_{X_i}(x, I_i(a))$ for $x \in X_i$ and $a \in A$. We claim that h_i is an isometric embedding. Because for $x_1, x_2 \in X_i$, we have

$$\begin{aligned}\|h_i(x_1) - h_i(x_2)\| &= \sup |d(x_1, I_i(a)) - d(x_2, I_i(a))| \\ &\leqq d(x_1, x_2).\end{aligned}$$

On the other hand, we can choose $I_i(a)$ arbitrarily close to x_i and hence

$$\sup |d(x_1, I_i(a)) - d(x_2, I(a))| \geqq d(x_1, x_2).$$

Thus we are done.

Let K be the subset $\{f \in F \mid |f(a_m)| \leqq 2R$ for $a_m \in A_1$, and $|f(a_{m_1 .. m_{n-1}}) - f(a_{m_1 .. m_n})| \leqq \varepsilon_{n-1}$ for all $n \in \mathbb{N}\}$. Then K is compact. Since $d(I_i(a_{m_1 .. m_{n-1}}), I_i(a_{m_1 .. m_n})) < \varepsilon_{n-1}$, $|h_i(x)(a_{m_1 .. m_{n-1}}) - h_i(x)(a_{m_1 .. m_n})| < \varepsilon_{n-1}$ and hence $h_i(X_i)$ is contained in K. Thus there is a subsequence $\{i_1\} \subset \{i\}$ so that $\{h_{i_1} I_{i_1}(a_{m_1})\}$ converges for all $a_{m_1} \in A_1$ since A_1 is a finite set. Taking the same procedure again and again, we get a chain of subsequences $\{i_1\} \supset \{i_2\} \supset \ldots$ where $\{h_{i_n} I_{i_n}(a_{m_1 m_2 .. m_n})\}$ converges for all $a_{m_1 m_2 .. m_n} \in A_n$. Let $\{j\}$ be the diagonal sequence. Then $\{h_j I_j(a_m)\}$ converges for all $a_m \in A$, and hence let us define $\varphi(a_m)$ to be its limit and Y to be the completion of $\{\varphi(a_m) : a_m \in A\}$. Then obviously X_j converges to Y.

We go back to the notation of § 2. Let ρ_t $(0 \leq t < t_0)$ be the deformation of hyperbolic structures constructed in § 2. We use C_t to denote a hyperbolic cone-manifold at time t and we simply denote (X_{C_t}, Σ_{C_t}) by (X_t, Σ_t). We wish to investigate what happens when $t \to t_0$. Since C_t is not a riemannian but complete metric space, we may apply Gromov's theory to this situation. Let $\{t_n\}$ be a positive increasing sequence which converges to t_0 and denote C_{t_n}, $X_{C_{t_n}}$ and $\Sigma_{C_{t_n}}$ again simply by C_n, X_n and Σ_n respectively.

Proposition 3.6 : Let x_n be an arbitrary point in C_n. Then there is a subsequence $\{k\} \subset \{n\}$ so that (C_k, x_k) converges to a complete metric space (C^*, y).

Proof. We use Gromov's criterion. We may assume that $x_n \notin \Sigma_n$ since we may change x_n slightly. Let P_n be a starshaped fundamental domain of C_n. Namely P_n is an expansion in $\mathbb{H}^3$ along the shortest geodesic segments from x_n. Then

$$\begin{aligned} N(\varepsilon, B_R(x_n, C_n)) &\leq N(\varepsilon, B_R(x_n, P_n)) \\ &\leq N(\varepsilon, B_R(x_0, \mathbb{H}^3)) \end{aligned}$$

The last term is bounded by a constant which depends only on ε and R. Hence we are done by Theorem 3.5.

§ 4 Injectivity radius and when it is uniformly bounded

The purpose of this section is to define injectivity radius for the hyperbolic cone-manifold in general and to see that C_n must have a thin part for sufficiently large n. More strictly, we show that if $\{C_n\}$ has uniform lower bound of injectivity radius, then it converges to a hyperbolic 3-cone-manifold, which contradicts the definition of terminal time.

Following the previous sections, we let C be a hyperbolic 3-cone-manifold, $\Pi = \pi_1(X_C - \Sigma_C)$ and ρ be its holonomy.

<u>Definition</u> (Volume of ρ) : Assume first that $\partial C = \phi$. Let $B = X_C - \Sigma_C$ and let $p : \tilde{B} \to B$ be the universal covering. Each element α of Π acts on $\tilde{B}$ as a covering translation T_α. Denote the fiber product $\tilde{B}\times\mathbb{H}^3/(T\times\rho)(\Pi)$ by E. Then the map $q : E \to B$ defined by $q([x,y]) = [x]$ $(x \in \tilde{B},\ y \in \mathbb{H}^3)$ becomes a fiber bundle of fiber $\mathbb{H}^3$ with the structure group $= \mathrm{Isom}^+\mathbb{H}^3$. Let $D : \tilde{B} \to \mathbb{H}^3$ be a developing map. Consider a section $s_0 : B \to E$ which makes the diagram below commute

$$\begin{array}{rcl} \mathrm{id}\times D : \tilde{B} & \longrightarrow & \tilde{B}\times\mathbb{H}^3 \\ \Big\downarrow /T & & \Big\downarrow /T\times\rho \\ s_0 : B & \longrightarrow & E\,. \end{array}$$

This certainly exists since $D\cdot T_\alpha = \rho_\alpha \cdot D$. Define $\mathrm{vol}_{s_0}(\rho)$ to be $\int_B s_0^*(dv)$ where dv is a volume form of a fiber $\mathbb{H}^3$ of E. Notice that $\mathrm{vol}_{s_0}(\rho) = \mathrm{vol}(B) = \mathrm{vol}(C)$. In case $\partial C \neq \phi$, define $\mathrm{vol}_{s_0}(\rho)$ to be a half of $\mathrm{vol}_{ds_0}(d\rho)$ where $d\rho$ and ds_0 correspond to the double of (X_C, Σ_C).

Let us think of another section $s_1 : B \to E$ which still satisfies the condition (A) : let U be a fundamental domain of B

in $\tilde{B}$ and think of s_1 as a map of U to $E|_U = U \times H^3$, then $pr \cdot s_1(x) = D(x)$ for $x \in X_C - \Sigma_C$ which is very close to Σ_C, where pr is the second projection : $U \times \mathbb{H}^3 \to \mathbb{H}^3$. Then again by the same formula, we can define the volume $\mathrm{vol}_{s_1}(\rho)$ with respect to s_1. However

<u>Claim</u> : $\mathrm{vol}_{s_1}(\rho) = \mathrm{vol}_{s_0}(\rho)$.

<u>Proof</u>. Since $\mathbb{H}^3$ is contractible, there is a homotopy s_t connecting s_0 and s_1 such that s_t stays constant on a neighborhood of Σ_C. Then

$$\begin{aligned} \mathrm{vol}_{s_0}(\rho) - \mathrm{vol}_{s_1}(\rho) &= \int_B s_0^*(dv) - \int_B s_1^*(dv) \\ &= \int_{\partial(B\times I)} s_t^*(dv) \\ &= \int_{B\times I} ds_t^*(dv) = 0. \end{aligned}$$

Thus volumes of ρ with respect to a section are identical as long as a section satisfies the condition (A).

<u>Proposition 4.1</u> : $\mathrm{vol}(C_t)$ <u>is uniformly bounded on</u> $0 \leq t < t_0$.

<u>Proof</u>. Since $\mathrm{vol}(C_t)$ is a continuous function on t, we only need to check boundedness near t_0. We may further assume that C_t has no euclidean ends for $t_0 - \varepsilon < t < t_0$ with sufficiently small ε because of our assumption that O has no boundary. We may also assume by taking ε sufficiently small that the topological type of (X_t, Σ_t) stays constant. Let us denote this topological space by (X, Σ). Then there are homeomorphisms h_t : $(X, \Sigma) \to (C_t, \Sigma_t)$.

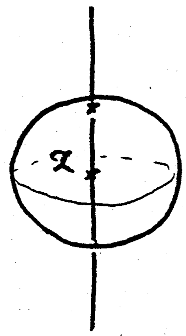
Take a triangulation K of (X, Σ) so that the 1-skeleton contains Σ. We further take a second barycentric subdivision K'' of K. Then the intersection of Σ and a 3-simplex of K'' is connected. Let us take $h_t : (X, \Sigma) \to (X_t, \Sigma_t)$ so that for a 3-simplex τ which intersects with Σ, $h_t(\tau)$ is a hyperbolic 3-simplex, and let K_t be a topological triangulation $h_t(K'')$.

We now define a section $s_t : B_t \to E_t$ to compute volumes. Let $D_t : \tilde{B}_t \to H^3$ be a developing map and let $U_t \subset \tilde{B}_t$ be a fundamental domain. Think first of a 3-simplex of K_t which does not intersect with Σ_t. It can be lift to $\tilde{C}_t$. Since we can choose U_t simplicially with respect to the triangulation K_t, we further can assume that it is lift to U_t. For such a 3-simplex, define s_t so that $pr \cdot s_t$ becomes a straight map in $\mathbb{H}^3$ fixing 4 vertices. Think next of a 3-simplex Δ which intersects with Σ_t. Since such a simplex is already hyperbolic by the definition of h_t, s_t can be canonically defined so that $pr \cdot s_t|_{\Delta - \Sigma_t} = D|_{\Delta - \Sigma_t}$.

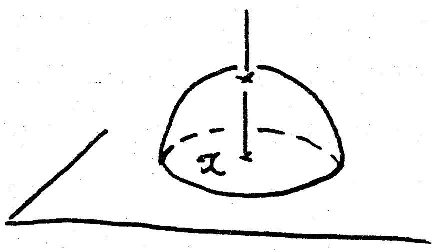
By definition, s_t satisfies the condition (A). Hence $\mathrm{vol}(C_t) = \mathrm{vol}_{s_t}(\rho_t) \leqq nv_3$, where $n = \#$ 3-simplices of K'' and v_3 is the volume of a regular ideal 3-simplex.

<u>Definition</u> : An ε-ball $B_\varepsilon(x)$ in a hyperbolic 3-cone-manifold is standard if it is one of the followings,

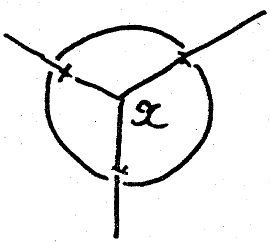
(1) around axis of elliptic,



(2) x on the boundary,



(3) x is a vertex of Σ.



<u>Definition</u> : Let C be a riemannian 3-cone-manifold. Then the injectivity radius at $x \in C$, denoted by $\mathrm{inj}(x)$, is the supremum of ε so that there is an ε-standard ball containing x. Such a ball may not have x as a center. Do not confuse ! For $\varepsilon > 0$, define the ε-thick part, $C_{\mathrm{thick}(\varepsilon)}$, to be all the union of standard ε-balls in C and the ε-thin part, $C_{\mathrm{thin}(\varepsilon)}$, to be the complement of the thick part.

Again go back to our situation. Recall that t_n is a positive increasing sequence which converges to t_0 and $\{C_n\}$ is a corresponding sequence of hyperbolic 3-cone-manifolds which converges to a metric space C^* in Gromov's sense. Notice that C^* is not a hyperbolic 3-cone-manifold with $\Delta_{C^*} = (X_0, \Sigma_0)$ because of Theorem 2.4, 2.6 and the definition of t_0.

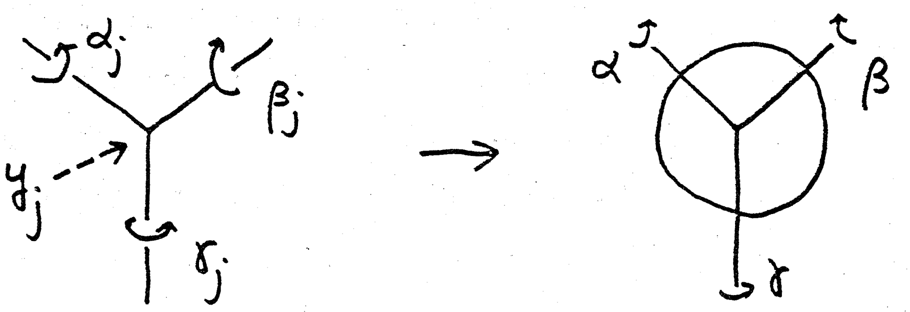
<u>Theorem 4.2</u> : <u>If there is an</u> $\varepsilon > 0$ <u>so that</u> $C_{n,\mathrm{thin}(\varepsilon)} = \phi$

<u>for all</u> n, <u>then there are a subsequence</u> $\{k\} \subset \{n\}$ <u>and a hyperbolic cone-manifold</u> C^* <u>with</u> $\Delta_{C^*} = (X_0, \Sigma_0)$ <u>such that</u> C_k <u>converges to</u> C^* <u>and also</u> ρ_k <u>converges to</u> ρ^* <u>which is the holonomy of</u> C^*. <u>In particular, this cannot happen</u>.

<u>Proof</u>. By the construction of C_n, there are a uniform lower bound of cone angles and a uniform lower bound of $2\pi-(\alpha+\beta+\gamma)$ where α, β, γ are cone angles around a vertex. Thus there exists a nonzero uniform lower bound of the volume of $\varepsilon/2$-balls in C_n. On the other hand, we have $\mathrm{vol}(C_t) < \kappa$ by Proposition 4.1. Let m_n be a number of $\varepsilon/2$-balls which pack C_n. Then by taking ε-balls with the same centeres, we get an open cover and hence we have $N(\varepsilon, C_n) \leqq m_n$. This implies that $N(\varepsilon, C_n)v_0 \leqq m_n v_0 \leqq \mathrm{vol}(C_n) \leqq \kappa$, and hence $N(\varepsilon, C_n) \leqq \kappa/v_0$. This shows that $\mathrm{diam}(C_n) \leqq 2\varepsilon\kappa/v_0$.

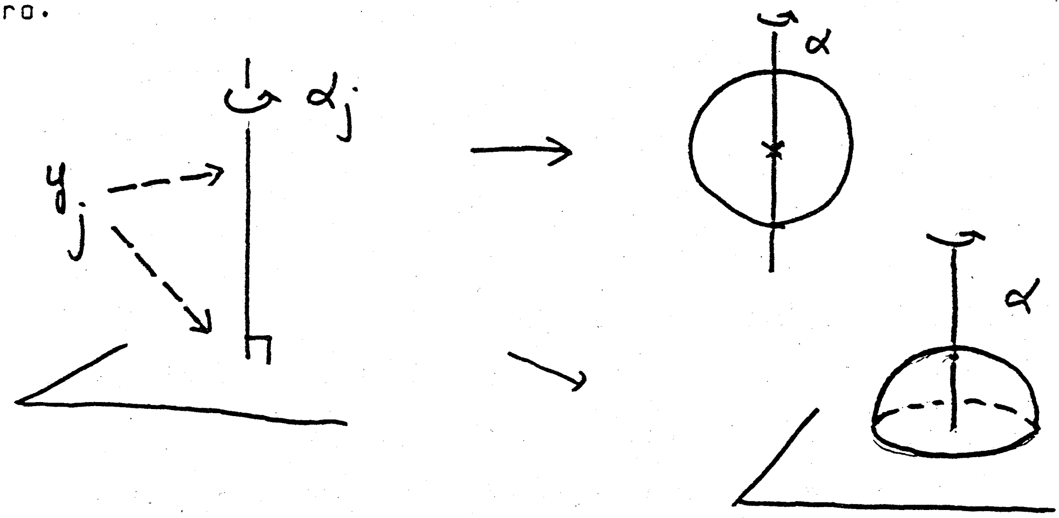
We now come to the situation where we can use Proposition 3.5. Taking a sequence $x_n \in \Sigma_n$, we get a subsequence $\{k\} \subset \{n\}$ which makes (C_k, x_k) converge to (C^*, x^*). Then since $\mathrm{diam}(C_n) \leqq R$, $\mathrm{diam}(C^*) \leqq R$. This shows that $B_R(x^*, C^*) = C^*$ and hence C^* turns out to be compact by Proposition 3.2. Furthermore C^* dose not depend on the choice of a sequence x_n by Proposition 3.1. We let R_k be an ε_k-approximation between C_k and C^* where $\varepsilon_k \to 0$ when $k \to \infty$, and for any $y \in C^*$, we let y_k be a point of $R_k^{-1}(y)$. Then for any $\delta > 0$, there is an ε_k-approximation $R_k | B_\delta(y_k)$ between $B_\delta(y_k)$ and $B_\delta(y)$.

<u>Case 1</u> : There is a subsequence $\{j\} \subset \{k\}$ so that $d(y_j, v_j) \to 0$ where v_j's are vertices of Σ_j corresponding to some vertex $v \in \Sigma_0$. If we let α_j, β_j and γ_j be cone angles around v_j in C_j, then they must converge to α, β and γ respectively and $B_\delta(y_j)$ approaches a standard 3-cone-ball illustrated below,



Since the limit is unique, $B_\delta(y)$ must be isometric to this standard 3-cone-ball.

Case 2 : There is a subsequence $\{j\} \subset \{k\}$ so that $d(y_j, e_j) \to 0$ when $j \to \infty$ where e_j's are edges of Σ_j corresponding to some edge $e \subset \Sigma_0$. By the same argument as in the case 1, $B_\delta(y)$ must be isometric to a standard hyperbolic 3-cone-ball or a half 3-cone-ball according to whether $\lim d(y_j, \partial C_j)$ is positive or zero.



Case 3 : There is a constant ι such that $d(y_k, \Sigma_k) > \iota$ for all k. Let $\eta = \min(\delta, \iota)$, then $B_\eta(y_k)$ is isometric to a possibly truncated standard hyperbolic ball. Hence $B_\eta(y)$ also must be a possibly truncated standard hyperbolic ball, and in particular, y is in a standard 3-ball or a half 3-ball.

Thus we have proved that C^* is a hyperbolic 3-cone-manifold. Moreover we have shown that the relation R_k corresponds the singular locus Σ_k to Σ_{C^*} and the boundary ∂C_k to ∂C^*. We then show that (X_{C^*}, Σ_{C^*}) is homeomorphic to (X_k, Σ_k) for sufficiently large k and in particular, $\Delta_{C^*} = (X_0, \Sigma_0)$. Take a hyperbolic triangulation K of C^* so that Σ_{C^*} is contained in the 1-skeleton of K. Then every four vertices which span a 3-simplex is locally in general position. Since C^* is compact, there are only finitely many vertices. Define a homeomorphism of {vertices of K} to C_k by corresponding a vertex v to any $w \in C_k$ which is related to v by R_k. Since {vertices of K} is a finite set, the image of four vertices which form a simplex by this map is still locally in general position for sufficiently large k. Thus they span locally a hyperbolic simplex in C_k and we get a hyperbolic triangulation of (X_k, Σ_k) which are combinatorially equivalent to K. In particular, (X_{C^*}, Σ_{C^*}) is homeomorphic to (X_k, Σ_k). This completes the proof.

§ 5 Noncompact euclidean 3-cone-manifolds

When thin part does not vanish, we must look at what happens when $\mathrm{inj}(x_n) \to 0$. To see this, we look at the geometric limit by rescalring its metric by $1/\mathrm{inj}(x_n)$. In this case, the metric in the limit becomes flat. If furthermore $\mathrm{inj}(x_n)/\mathrm{diam}(C_n) \to 0$, then as we will see later, it becomes noncompact euclidean 3-cone-manifold with cone angle $\leq \pi$. The purpose of this section is to classify those.

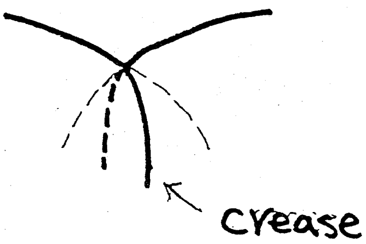
Let E be a noncompact euclidean 3-cone-manifold with cone angle $\leq \pi$ and p be a base point in E. Then let P_p be a a starlike fundamental domain $\subset \mathbb{E}^3$ centered at p.

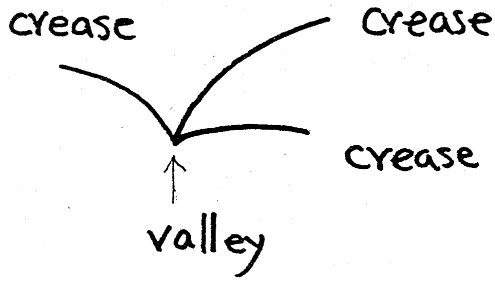
<u>Claim</u> : P_p is convex.

<u>Proof</u>. Exercise !

Let $B_r(p)$ be an r-ball centered at p and let S_r be its boundary $\partial B_r(p)$. We call $\text{area}(S_r)/r^2$ the visual area of S_r. Let $A(t)$ be a visual area of $S_{\exp(t)}$. Then it is monotone nonincreasing function since P_p is convex.

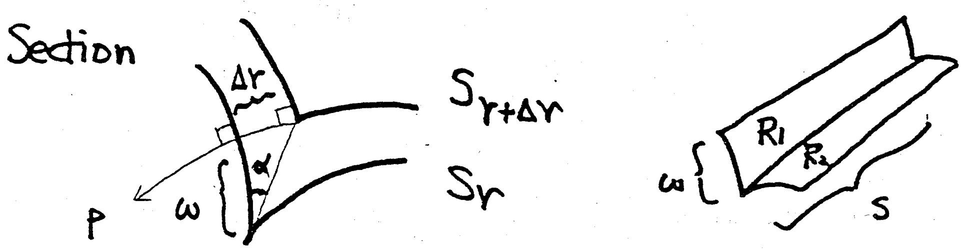
We call the following singularity a crease and their intersection point a valley.





We will see how the crease and valley contribute to $A(t)$.

<u>Case 1</u> : Crease. Look at the picture below.



We can see that $\tan\alpha$ approaches $\Delta r/w$ when $\Delta r \to 0$. Let s be the length of a crease in question. Then area R_1 = area R_2 = area R and we have

$$\mathrm{Area}(S_{r+\Delta r}) = (\mathrm{area}(S_r) - 2\ \mathrm{area}\ R)(r+\Delta r)^2/r^2.$$

If we let $V(r) = \mathrm{area}(S_r)/r^2$,

$$V(r+\Delta r) - V(r) = -2\ \mathrm{area}(R)/r^2.$$

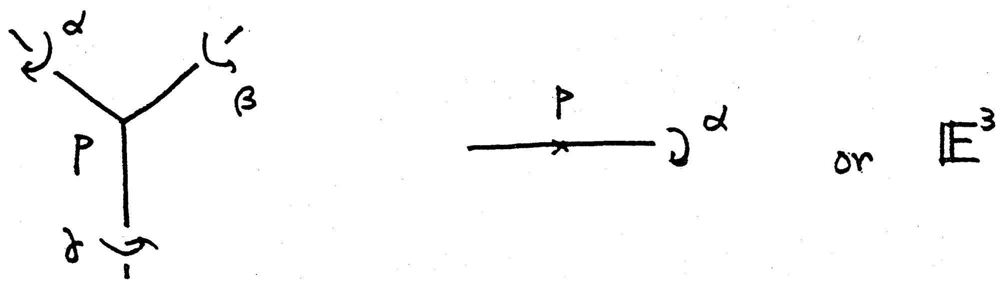
On the other hand, area(R) is approximately equal to sw and hence $s\Delta r \cot\alpha$. That implies the derivative $dV(r)/dr = -2s\cot\alpha/r^2$. Since $A(t) = V(r)$ and $r = \exp t$, we have

$$\mathring{A}(t) = -dV(r)/dy\cdot dr/dt = -2s\cot\alpha/r.$$

<u>Case 2</u> : Valley. Again since $V(r+\Delta r) - V(r) = -\mathrm{area}(R)/r^2$, area(R) has the same order as $(\Delta r)^2$. Hence $\dot{V}(r) = 0$ and the valley does not contribute the change of $A(t)$.

We have shown by these observation that $A(t)$ is differentiable except when the crease is newly produced.

<u>Proposition 5.1</u> : <u>If</u> A <u>is constant for all</u> t, <u>then</u> E <u>is</u>



<u>Proof</u>. Since the visual area of S_r is constant, the crease cannot be produced in the process. This means that S_r intersects with neither itself nor a singular locus. Hence the only possible euclidean cone-manifold which possesses such a base point is the above three.

In what follows, we are concerned with the behavior of $A(t)$ as $t \to \infty$ to see what the end of E looks like.

Let P be a (non geodesic) n-gon in $\mathbb{E}^3$, K a Gaussian curvature, dA an area form, k_g a geodesic curvature of ∂P. Then by Gauss Bonnet, we have

$$-\int_P K\, dA = \int_{\partial P} k_g\, ds + (n-2)\pi - \sum_{i=1}^{n} \alpha_i ,$$

where α_i is an angle of the i-th vertex of P. This is the general formula.

Let us now divide S_r by n_i-gons P_i, $1 \leqq i \leqq f$, so that each crease is contained in some ∂P_i and each valley is contained in the vertices of some P_i. Let Γ be a 1-complex which consists of $\bigcup \partial P_i$. Denote #{edges of P_i} by m_i and #{edges of Γ} by m. Then $m = \sum m_i/2$. Also denote #{vertices of Γ} by n. At each vertex v_j of Γ, finitely many P_i's meet. Let us denote them, say, by $P_{j_1}, \ldots P_{j_k}$ and also denote each angle of P_{j_i} at v_j by α_{ji}. $\alpha_j = \alpha_{j1}+\cdots+\alpha_{jk}$ is the total angle centered at v_j. It is obvious that $\sum_j \alpha_j$ ie equal to the summation of all the total angles of P_i's at vertices. Also notice that each α_j is 2π unless v_j is a valley.

Again by Gauss Bonnet,

$$-\sum_{i=1}^{f} \overbrace{\int_{P_i} K_i\, dA_i}^{I_1} = \sum_{i=1}^{f} \overbrace{\int_{\partial P_i} k_g^{(i)} ds}^{I_2} + \overbrace{\sum_{i=1}^{f} (m_i-2)\pi - \sum_{j=1}^{n} \alpha_j}^{I_3} .$$

We interpret each terms. Since P_i is on S_r, $K_i = 1/r^2$ and hence

$$I_1 = -\mathrm{area}(S_r)/r^2 = -A(t).$$

Think of I_2 next. On each nonsingular edge of Γ where P_i and P_j meet, $k_g{}^{(i)}$ and $k_g{}^{(j)}$ are cancel out. Thus the integral needs only on the edges which are crease. Let $e_1, \ldots, e_q$ be edges of Γ which are crease and let $s_1, \ldots, s_q$ be their length. Since the crease is on the circle of radius $r \sin\alpha$, we have $|r''| = 1/r \sin\alpha$. Thus $k_g(s) = -\cos\alpha\,|r''| = (-1/r)\cot\alpha$ and

$$I_2 = \sum_{i=1}^{q} \int_{e_i^+ \cup e_i^-} k_g{}^{(i)} = -\sum_{i=1}^{q} \frac{2s_i}{r} \cot\alpha = \dot{A}(t).$$

Finally we have

$$\begin{aligned} I_3 &= \sum_{i=1}^{f} (m_i - 2)\pi - \sum_{j=1}^{n} \alpha_j \\ &= (2m-2f)\pi - 2n\pi + \sum_{j=1}^{n} (2\pi - \alpha_j) \\ &= -2\pi\chi(S_r) + \sum_{j=1}^{n} (2\pi - \alpha_j) . \end{aligned}$$

The following is the derivative formula we have obtained.

$$\text{(A)} \qquad \dot{A}(t) = 2\pi\chi(S_{\exp(t)}) - A(t) - \sum_{j=1}^{n} (2\pi - \alpha_j).$$

<u>Remark</u> : Choose a base point $p \in E$. For a point $x \in \partial P_p$, we let ν_x be the ray which runs from p to x. If $d(p,x) \to \infty$, then ∂P_p and ν_x becomes gradually parallel. Simultaneously S_r and ∂P_p intersect gradually perpendicularly each other.

Think of a polygonal division of S_r naturally arisen in the fundamental domain. Let P be a fundamental polygon of S_r which is contained in P_p. We have by Gauss Bonnet that

$$0 > -\int_P K\, dA = \int_{\partial P} k_g ds + (n-2)\pi - \sum_{i=1}^{m} \theta_i ,$$

where m = # of vertices of P and θ_i is an angle of the i-th vertices. We claim that the first term approaches zero when $t \to \infty$. This is because k_g approaches zero with order r^{-2} however the length of ∂P approaches ∞ with order r. Hence we have that for any $\varepsilon > 0$, there exists t_0 so that

$$\sum_{i=1}^{n} (\pi - \alpha_i) < 2\pi + \varepsilon.$$

When $\alpha_{j1}, \ldots, \alpha_{jk}$ meet at valley v_j on $S_r \subset E$ where $\alpha_{j1} + \ldots + \alpha_{jk} \leqq 2\pi$, we must have

$$\sum_{i=1}^{k} (\pi - \alpha_{ji}) = k\pi - (\alpha_{i1} + \ldots + \alpha_{1k}) \geqq (k-2)\pi.$$

Thus if we take t_0 sufficiently large, then the summation of the left hand side in terms of j must be less than $2\pi+\varepsilon$ by the previous inequality. That means k can be at most 4 and moreover if $k = 4$, then $j = 1$ which means there is only one valley, and if $k = 3$, then $j \leqq 2$ which means that the number of valley is less than three.

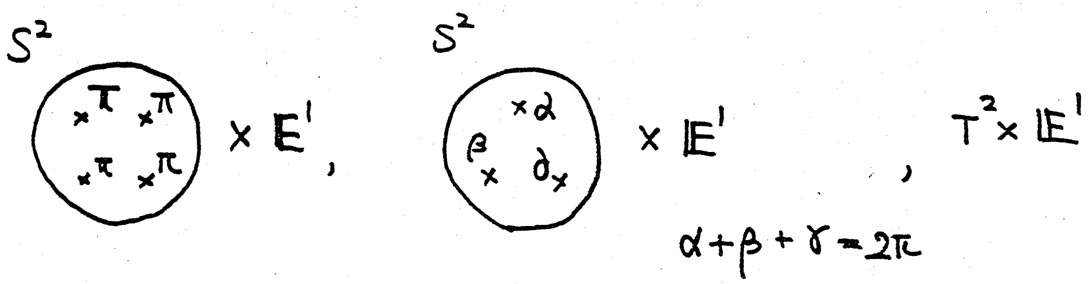
We rewrite the formula by

$$\dot{A}(t) = 2\pi\chi(S_{\exp(t)}) - A(t) - P(t) - Q(t),$$

where $P(t)$ is contribution of valleies which are not contained in Σ and $Q(t)$ is contribution of $S_r \cap \Sigma_E$. Notice that $\lim P(t) = 0$ $(t \to \infty)$. This is because when t goes to ∞, each crease approaches geodesics and hence the total angle around the valley, whose number is at most two, tends to 2π.

<u>Proposition 5.2</u> : E <u>has at most two ends</u>. <u>If</u> E <u>actually has two ends, then</u> E <u>is a product of a compact euclidean</u> 2-<u>cone</u>-

<u>manifold and a 1-dimensional euclidean space. More precisely</u> E <u>is one of the followings</u>.



<u>Proof</u>. For each end of E, we can choose a ray in P_p from p to infinity. So given two ends, we have two rays ν_1 and ν_2 which tend to each infinity. Suppose $\nu_1 \cup \nu_2$ is not a straight line, then since P_p is convex, a convex plane bounded by $\nu_1 \cup \nu_2$ is contained in P_p and we cannot have a compact face which separates given ends. This shows that E has at most two ends and if E actually has two ends, then corresponding half lines form a straight line in P_p.

Suppose E has two ends. Then since any face of P_p does not intersect with $\nu_1 \cup \nu_2$, it must be parallel to $\nu_1 \cup \nu_2$. Moreover since E actually has two ends, the faces of P_p must surround $\nu_1 \cup \nu_2$. Let Q be a 2-dimensional polygon which intersects perpendicularly to the faces. Then ∂Q must be glued with ∂Q via the identification of P_p to produce E because P_p is starlike and hence the identification does not contain the translation factor along $\nu_1 \cup \nu_2$. This proves the proposition.

In what follows, we will be exclusively concerned with E with only one end. In this case, $S_{\exp(t)}$ is connected if t is sufficiently large.

Proposition 5.3 : (1) There exists t_0 such that $\chi(S_{\exp(t)})$ is non negative and increasing for $t \geqq t_0$.

(2) If $\chi(S_{\exp(t)}) = 0$ for all $t \geqq t_0$, then there is no axis of Σ_E which tends to infinity.

(3) If there is $\bar{t} \geqq t_0$ so that $\chi(S_{\exp(\bar{t})}) = 2$, then the number of axis of Σ_E which tend to infinity is $\leqq 4$. Moreover if it is 4, then their cone angles all are equal to π.

Proof. Recall that we have a formula

$$\dot{A}(t) = 2\pi\chi(S_{\exp(t)}) - A(t) - P(t) - Q(t) ,$$

where the last three terms are non-negative.

First of all, $\chi(S_{\exp(t)})$ is constant when t is sufficiently large because then $\partial S_{\exp(t)}$ intersects with P_p in almost right angle and there is no chance to produce or to reduce genus. If $\chi(S_{\exp(t)}) < 0$, then $\dot{A}(t) <$ const < 0 for all $t > t_0$ and hence $A(t)$ becomes zero when t is sufficiently large. This means that E is compact which is out of our argument.

If $\chi(S_{\exp(t)}) = 0$, then $\dot{A}(t) \leqq - Q(t)$. If there is an axis which tends to infinity, then $A(t) \leqq -(2\pi-\alpha) \leqq -\pi$ where α is its cone angle. then again by the same reason, E becomes compact which is out of our argument.

Assume that we have the condition of (3). In general $\lim P(t) = 0$ $(t \to \infty)$. Since E is noncompact, there is a diverging sequence $\{t_n\}$ so that $\lim \dot{A}(t_n) = 0$. For this sequence, we have $Q(t_n) < 4\pi+\varepsilon$ ($\varepsilon \to 0$ when $n \to \infty$). Let $e_1, \dots, e_k$ be edges of Σ_E which tend to infinity. The summation of angles around $e_j \cap S_{\exp(t_n)}$, denoted by α_j, is obviously more than the cone angle ι_j of e_j. Hence we have

$$4\pi+\varepsilon > Q(t_n) \geqq \sum_{j=1}^{k}(2\pi-\alpha_j) \geqq \sum_{j=1}^{k}(2\pi-\iota_j) \geqq 4\pi .$$

Hence $k \leqq 4$. Moreover if $k = 4$, then since $2\pi - \iota_j = \pi$, the cone angle of each e_j must be π.

We are now ready to classify noncompact euclidean 3-cone-manifolds with cone angle $\leqq \pi$.

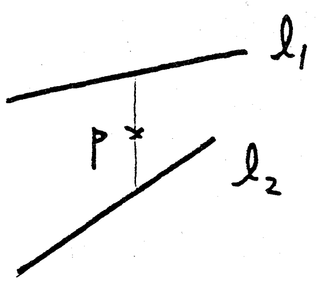
Case 1 : $\Sigma_E = \phi$. E is then a noncompact euclidean 3-manifold. Namely, $E = \mathbb{E}^3/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^3)$. When Γ is trivial, then $E = \mathbb{E}^3$. When Γ is cyclic, then E is $S^1 \times \mathbb{E}^2$ where the product may be twisted. For the rest case, there is an abelian group Γ' of Γ of finite index which are generated by translations. Since E is noncompact in our case, Γ' is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Also since $\Sigma_E \neq \phi$ and E has only one end, E must be a twisted product of a Klein bottle with the 1-dimansional euclidean space.

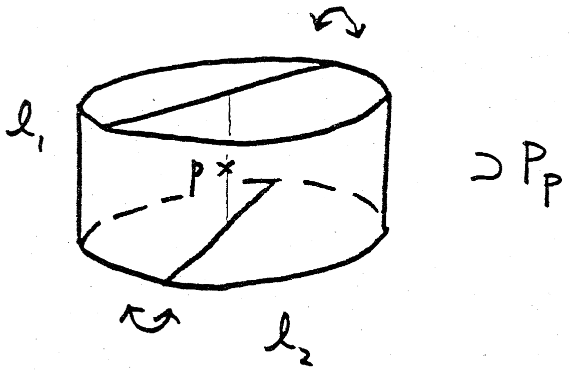
Case 2 : Σ_E consists of lines. Since there are at most four axis which tend to infinity by Proposition 5.3, Σ_E must consist of one or two lines. Also we may assume that there is an increasing sequence $\{t_n\}$ so that $S_{\exp(t_n)} = S^2$.

(1) When Σ_E consists of a line. Take a base point p on Σ_E. Then since an ε-ball is standard if ε is sufficiently small, we have $A(\log \varepsilon) = 4\pi(\alpha/2\pi) = 2\alpha$. On the other hand, we do have $\dot{A}(t_n) = 4\pi - A(t_n) - P(t_n) - Q(t_n)$. When $n \to \infty$, $\dot{A}(t_n) \to 0$, $P(t_n) \to 0$, $Q(t_n) \to 4\pi - 2\alpha$, and hence $A(t_n) \to 2\alpha$. This shows that A stays constant and we are done by Proposition 5.1.

(2) When Σ_E consists of two lines ℓ_1 and ℓ_2. The cone angles both must be π by Proposition 5.3. Let p be the center of the shortest path connecting ℓ_1 and ℓ_2. Since their cone angles are π, P_p must be contained in the following picture in

general.

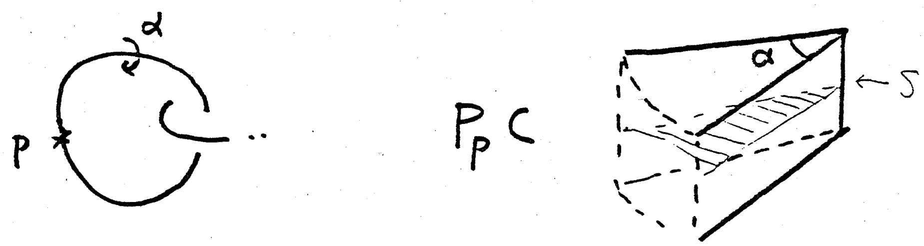




Since other face of P_p does not intersect either ℓ_1 or ℓ_2, P_p is actually equal to the picture above if ℓ_1 is not parallel to ℓ_2. When P_p does not have another face, then P_p is still the same even if ℓ_1 is parallel to ℓ_2.

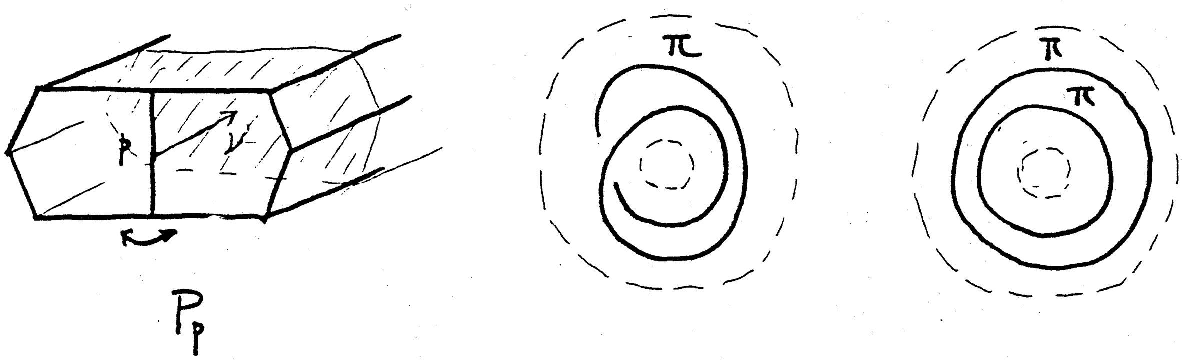
Thus assume that P_p has another face and ℓ_1 is parallel to ℓ_2. Let $Q \subset P_p$ be a hyperplane cut which intersects vertically with ℓ_1 and ℓ_2 and which contains p. Then by the same reason as before, ∂Q is identified with itself by the identification to produce E. Let Q' be a 2-cone-manifold obtained from Q by identifying edges containing $Q \cap \ell_1$ and $Q \cap \ell_2$. Q' is still convex and hence Q must be a rectangle and further more each vertex must have angle $\pi/2$. Therefore Q becomes P^2 with two cones of angle π after identfication. This implies that E is a twisted product of P^2 with two cones of angle π and the 1-dimensional euclidean space.

<u>Case 3</u> : Σ_E consists of circles. Take a base point p on a circle of Σ_E with cone angle α. Since P_p is divided at the antipodal point of p on the circle, it is contained in the picture below.



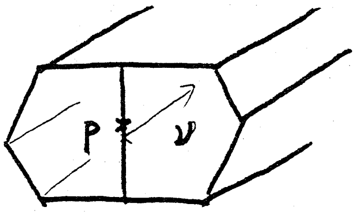
When P_p does not have another face, then E must be the product of a circle with a 2-dimensional open disk with a cone of angle α.

Suppose that P_p has another face. If $\alpha \neq \pi$, then since a hyperplane cut S illustrated in the picture becomes compact by convexity of P_p, E itself also must be compact. This is out of our argument. Thus we assume that $\alpha = \pi$. Let ν be a ray from p to infinity. P_p is still convex after identifying faces along singular locus containing p and hence another face must be parallel to ν. Since a hyperplane cut which is vertical to these faces has generically no intersection with a singular locus, it turns out to be either a torus or a Klein bottle after final identification. Hence we can have the following two cone-manifolds as E.

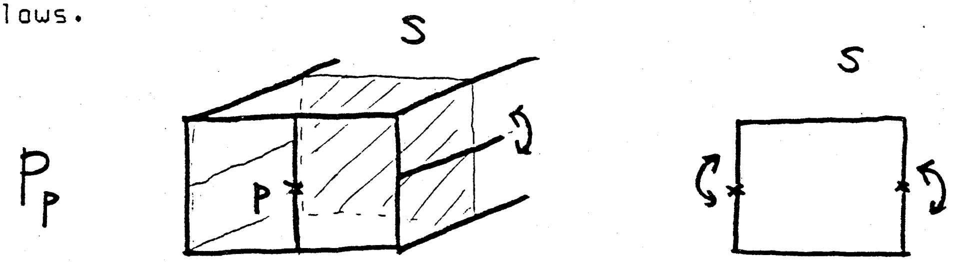


<u>Case 4</u> : Σ_E consists of the union of circles and lines and hence it has no vertices. Take a base point p on the circle of

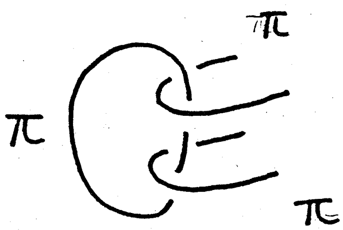
Σ_E. Then we do have the same picture as in the case 3.



By Proposition 5.3, the number of axis of Σ_E which tend to infinity is 2 or 4. Suppose we have two such. If we rechoose a base point p on this line, then by the same argument as in the case 2, $A(t)$ stays constant and Σ_E cannot have circle. Thus assume that we have 4 axis which go to infinity. Again by Proposition 5.3, their cone angles all must be π. Since they are on the face of P_p, they are parallel to ν. This shows that a hyperplane cut S which does not contain p becomes S^2 with 4 cones of angle π after identification. We can draw the picture of P_p as follows.

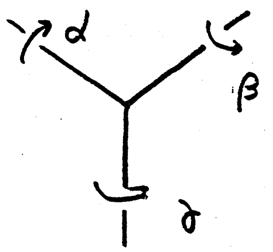


Hence E must be



<u>Case 5</u> : Σ_E has a unique vertex v and moreover each edge

from v goes to infinity. If there is another line which goes to infinity, then we get at least 5 axis which tend to infinity. This contradicts Proposition 5.3. Hence any other component of Σ_E is a circle. Let us take p to be v and let the cone angles around v be α, β, γ respectively. Then $A(t) = \alpha+\beta+\gamma-2\pi$ for sufficiently small t. On the other hand, as $t \to \infty$, S_t intersects only with three axis. Recall that there is a diverging sequence $\{t_n\}$ so that $\dot{A}(t_n) \to 0$ when $n \to \infty$ since E is noncompact. For such a sequence, we have $4\pi - A(t_n) - Q(t_n) \to 0$ when $n \to \infty$. Since $Q(t_n) \to (2\pi-\alpha)+(2\pi-\beta)+(2\pi-\gamma)$, $\lim A(t_n) = \alpha+\beta+\gamma-2\pi$. This shows again that $A(t)$ stays constant since $A(t)$ was monotone nonincreasing. Hence E must be



by Proposition 5.1.

<u>Proposition 5.4</u> : <u>Let</u> Q <u>be a spherical</u> 2-<u>cone</u>-<u>manifold</u> $S^2(\alpha,\beta,\gamma)$.

(1) <u>If</u> $\alpha, \beta, \gamma \leq \pi$, <u>then its diameter is</u> $\leq \pi$. <u>In other words</u>, $d(p, q) \leq \pi/2$ <u>for any</u> $p, q \in Q$.

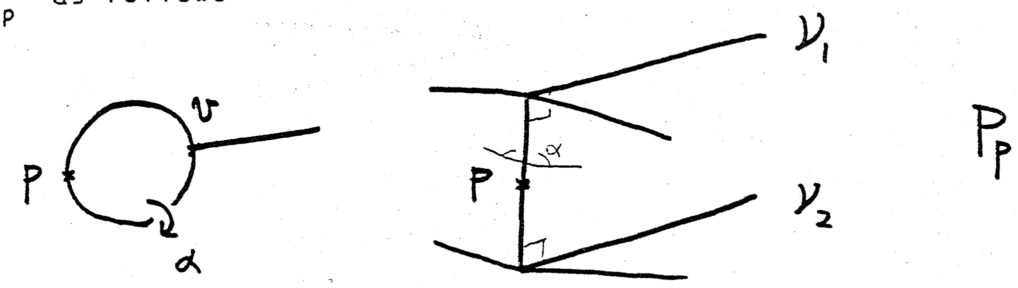
(2) <u>If furthermore</u> $\alpha, \beta < \pi$, <u>then its diameter is</u> $< \pi$. <u>That is to say</u>, $d(p, q) < \pi$ <u>for any</u> $p, q \in Q$.

<u>Here we assume that the curvature of</u> Q <u>is</u> 1.

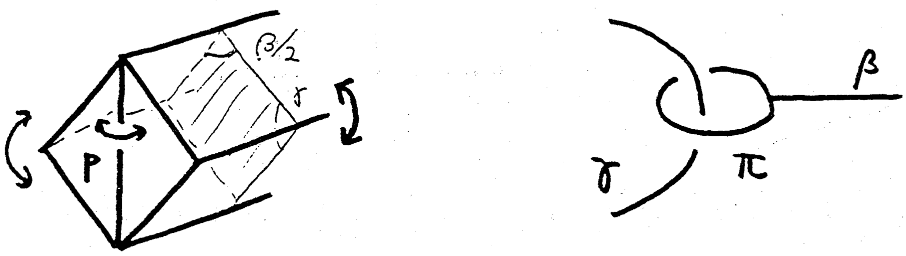
<u>Proof</u>. Develop Q on S^2 and enjoy your spherical geometry !

Case 6 : Otherwise.

(1) If Σ_E contains an edge which is closed at the vertex v. The other edge from v must go out to infinity. Take a base point p on the center of the closed circle and we get a fundamental dmain P_p as follows.



The lines ν_1 and ν_2 in the picture, which come down to the other edge from v, must intersect perpendicularly to the edge containing p since otherwise P_p becomes compact. Also ν_1 must be parallel to ν_2 again since otherwise P_p becomes compact. Suppose that $\alpha < \pi$. Then it produces a cone of angle $> 2\pi$ after identification. This is contradiction and hence $\alpha = \pi$. Thus P_p is the product of a fundamental domain of a 2-cone-manifold and $[0,\infty)$. A hyperplane cut which does not contain p becomes an euclidean 2-cone-manifold and hence E must be

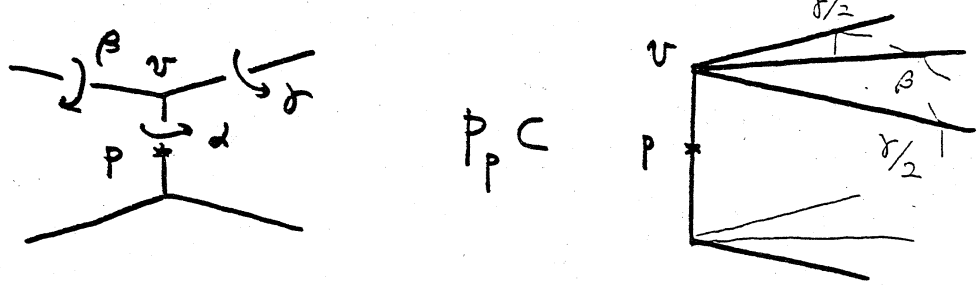


(2) Σ_E contains at least two vertices and does not contain and edge which is closed at a vertex.

Claim : There is an edge which connects two vertices.

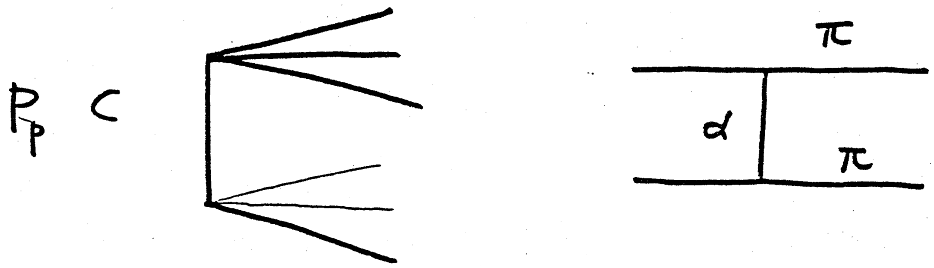
<u>Proof</u>. Otherwise either there are 6 axis which go to infinity.

Take a base point p on the center of the edge in the claim. Then we have a picture of P_p as follows.

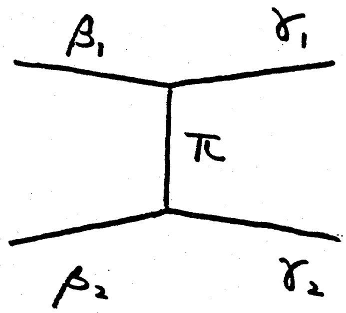


If β, $\gamma < \pi$, then applying Proposition 5.4 to an ε-sphere centered at v, we have that $\delta < \pi/2$. This implies that P_p is compact, which is out of our argument. Thus we may assume that $\beta = \pi$. Suppose $\alpha < \pi$, then again by Proposition 5.4, $\delta < \pi/2$ unless $\gamma = \pi$. Since the case $\delta < \pi/2$ is out of our argument, we may assume that $\gamma = \pi$.

(a) When $\alpha < \pi$ and $\beta = \gamma = \pi$: There cannot be another face. Thus E is

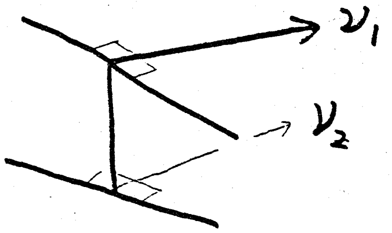


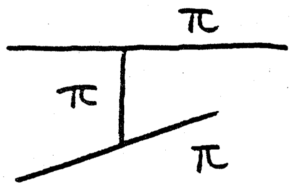
(b) When $\alpha = \pi$: Let us name the cone angles as follows



where either β_1 or γ_1 and also either β_2 or γ_2 are π.

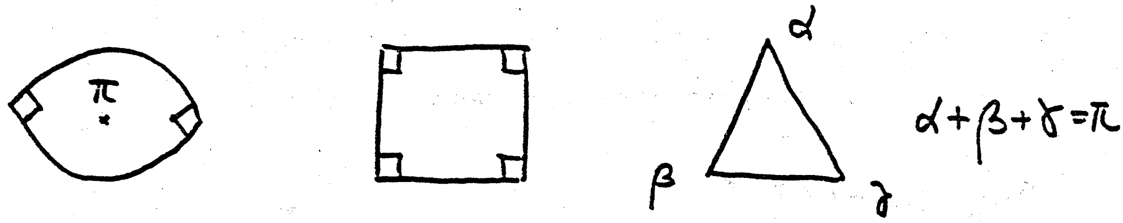
(b-1) when $\gamma_1 = \gamma_2 = \pi$. If ν_1 is not parallel to ν_2, then there cannot be another face and hence we have



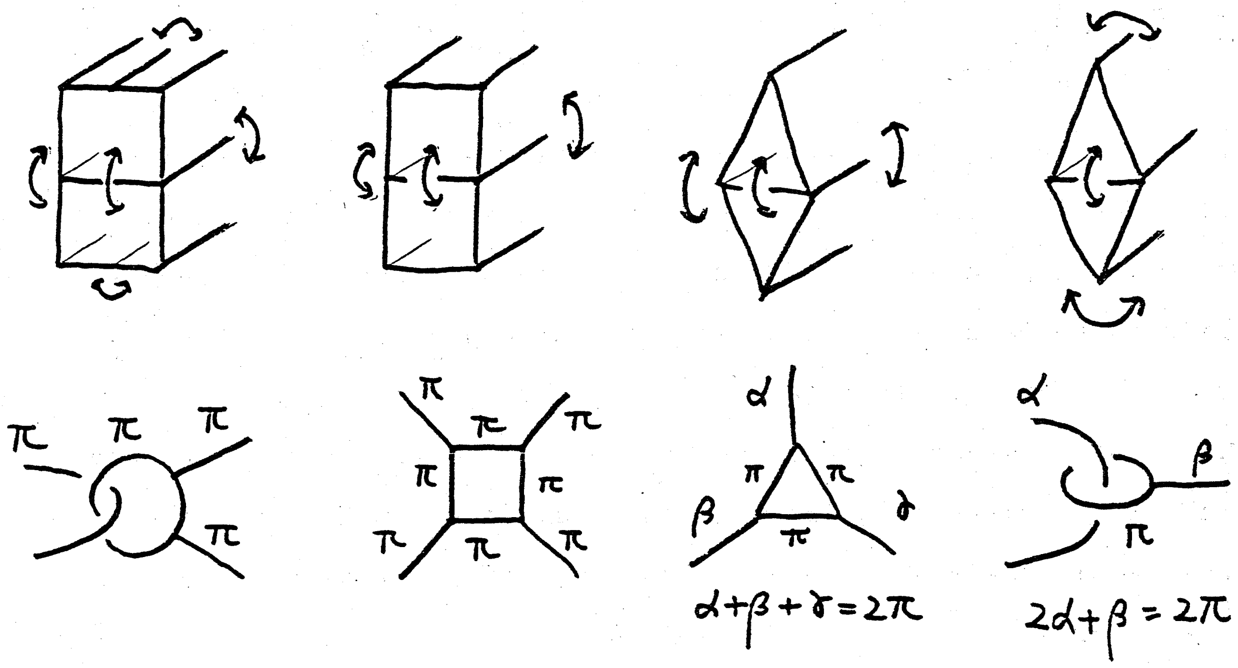


When ν_1 and ν_2 are parallel and there are no other face, the picture will be the same as above and we are done. Hence assume that ν_1 is parallel to ν_2 and with another face. Then such a face must be parallel to a hyperplane cut containing ν_1 and ν_2. Thus P_p turns out to be a product of a double of a compact euclidean 2-cone-manifold along an edge and $[0,\infty)$. Since the cone-manifolds we discuss in the next case have the same type of the fundamental domain, we leave a classification of this case by the end of (b-2).

(b-2) When $\gamma_1 < \pi$ $(\gamma_2 = \beta_1 = \pi)$. Then ν_1 must be parallel to ν_2. Because if not, and further if $\gamma_2 < \pi$, then E becomes compact, which is out of our argument. If $\gamma_2 = \pi$, then two lines which are the intersection of the roof of P_p containing ν_1 and the base of P_p, is not parallel to ν_2. Thus E becomes compact, which is again out of our argument. We have shown that ν_1 is parallel to ν_2. Thus P_p is a product of a compact half euclidean 2-cone-manifold along an edge and $[0,\infty)$. We now classify noncompact enclidean cone-manifolds having a fundamental domain of this type. First notice that there are three compact half euclidean 2-cone-manifolds illustrated below.



Thus the possible cases are the followings.



Summarizing the arguments above, we have

Theorem 5.5 : Let E be a noncompact euclidean 3-cone-manifold with cone angles $\leq \pi$. Then we can choose a fundamental domain P of E in the followings.

(1) A product of a compact euclidean 1-orbifold and a noncompact euclidean 2-cone-manifold.

(2) A product of a noncompact euclidean 1-orbifold and a compact euclidean 2-cone-manifold.

(3) One listed in Proposition 5.1.

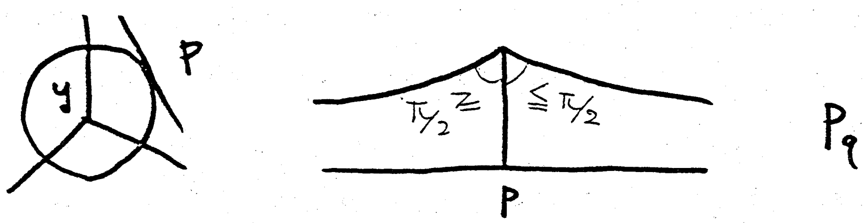
§ 6 Rescaling geometric limits

The following propostion is a key for analyzing injectivity radius of cone-manifolds.

Proposition 6.1 : *Let* C *be a riemannian 3-cone-manifold with constant curvature* $-1 \leq K \leq 0$. *Suppose there is a constant* $K_1 > 0$ *so that the angle of each elliptic axis is greater then* K_1 *and less than or equal to* π *and the sum of three angles of elliptic axis which meet at a vertex is greater than* $2\pi + K_1$. *Then given* R *and* $K_2 > 0$, *there is a constant* $\delta > 0$ *which depends only on* R, K_1, K_2 *and does not depend on* C *so that if* $inj(x) > K_2$, *then* $inj(y) > \delta$ *for any* $y \in B_R(x,C)$.

Proof. We split the argument into three cases. The first one is when y is a vertex of Σ. Let r_1 be $\sup\{ r \mid B_r(y)$ is standard $\}$. We then further split this case into two parts.

Case 1-a) When $\partial B_{r_1}(y)$ contacts some edge of Σ. Let p be this contact point and let q be a central point of the segment connecting p and y. Develop C centered at q. The following is an abstract picture of this fundamental domain P_q.

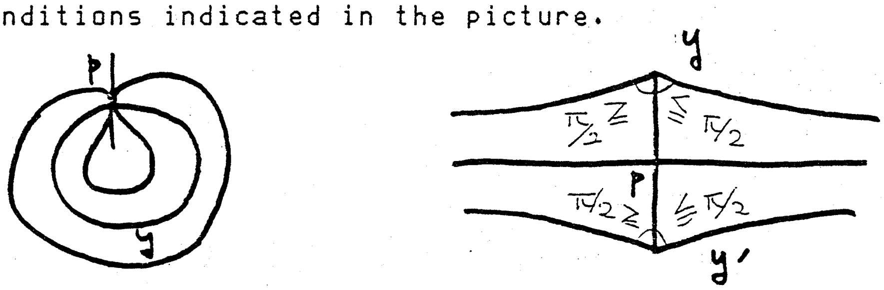


By Proposition 5.4, the both angles indicated in the picture are at most $\pi/2$. Now since $inj(x) > K_2$ by the assumption, there is a point x_0 in C so that $B_{K_2}(x_0)$ is standard and contains x. Think of a ball of radius r_1+R+2K_2 centered at q. This obviously contains $B_{K_2}(x_0)$ and we get the inclusion :

$$B_{r_1+R+2K_2}(q) \supset B_{K_2}(x_0)$$

The volume of the right hand side is bounded from below by some constant which depends only on K_1 since that is a standard ball. However the volume of the left hand side approaches zero when r_1 goes to zero. That is contradiction and there must be a lower bound of r_1 in terms of R, K_1 and K_2.

Case 1-b) When $\partial B_{r_1}(y)$ contacts itself. Let p be this contact point and develop C centered at p. Then we get the following abstract picture and again by Proposition 5.4, we have angle conditions indicated in the picture.



The same argument as in the above can be applied to this case.

We then discuss the second case when y is on the edge e_1 of Σ. Let r_2 be $\sup\{ r \mid B_r(y)$ is standard $\}$. We may assume that the distance between y and any vertex of Σ is bounded by a lower bounding constant s_1 of r_1 obtained in the previous case. That is to say, $r_2 \leqq s_1/2$. We further split this case into two parts.

Case 2-a) $\partial B_{r_2}(y)$ contacts an edge e_2 of Σ. Let p be this contact point and let q be the central point of the segment a_0 connecting y and p. Then develop C centered at q and denote it by P_q. Imagine that the face F of P_q containing e_2 is almost parallel to e_1 when the angle between e_1 and the segment connecting y and p is near $\pi/2$. Let $\theta(r_2)$ be the angle between 0 and $\pi/2$ so that $\cos\theta(r_2) = 2r_2/s_1$. Then $\theta(r_2)$ approaches $\pi/2$ when r_2 goes to zero.

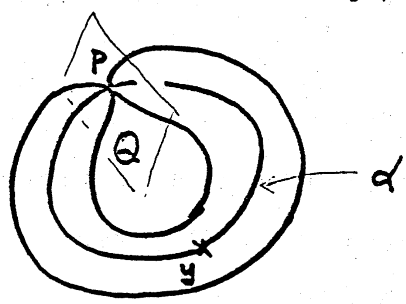
Suppose that there is a sequence of r_2 of this situation which converges to zero. Furthermore suppose corresponding angles θ's all are greater than $\theta(r_2)$'s. Then since F becomes almost parallel to e_1, the volume of $B_{R+2K_2+r_2}(q)$ approaches zero when r_2 goes to zero. This is again in contradiction with the inclusion of the standard ball. Thus r_2 has a lower bound and we are done. If corresponding angles θ's all are less than $\theta(r_2)$'s, then the argument will be a little bit complicated, however we can deduce to contradiction. Let a_1 be the shortest segment in $\mathbb{H}^3$ connecting e_1 and e_2. We are thinking that P_q is in $\mathbb{H}^3$. If there is no singular locus in the rectangle bounded by e_1, e_2, a_0 and a_1, then developing C centered at the midpoint q' of a_1, we get a ball $B_{r_2+R+K_2}(q')$ whose volume converges to zero when length $a_1 \leqq r_2$ goes to zero. This is in contradiction with the inclusion of a standard ball. Thus there must be a singular locus e_3 in the rectangle. We may assume that the distance between the intersection point of e_3 and the rectangle and either e_1 or e_2 is $\leqq r_2/2$. Choose one that satisfies the inequality and do the same argument as above to this locus and e_3. The situation is completely same as one at the beginning of the previous argument except that the distance between two edges is less than $r_2/2 < s_1/2$, which is a half of the original situation. Do the same argument again. This process terminates by finitely many steps and we find the shortest segment a connecting two singular loci in C within a small distance from y. More precisely, $d(y, a) < \sum_{k=1}^{\infty} s_1/2^k = s_1$. Thus by choosing the central point q of a and developing C centered at q, we get a ball $B_{2r_2+R+K_2}(q)$ whose volume converges to zero when r_2 goes to zero. This is in contradiction with the inclusion of a standard ball within a small distance from y.

We remark to clarify the situation that the length of a_i cannot be zero in the above process since if it happens, then e_i^{ℓ} and e_{i+1} have a common vertex which is contained in $B_{s_1}(y)$. That contradicts the assumption.

Case 2-b) When $\partial B_r(y)$ contacts itself. Let p be the contact point and develop C centered at p. We get the almost same fundamental domain P_p as in the previous case. Actually the same argument can be applied to this P_p.

Let s_2 be the minimum of the lower bounds obtained in the previous two cases. Then actually for any point y in the s_2-neighborhood of Σ, $\mathrm{inj}(y)$ is greater than s_2. The last case is thus the following.

Case 3) When y is in $(C - \mathcal{N}_{s_2}(\Sigma)) \cap B_R(x)$. Let r_3 be $\sup\{ r \mid B_r(y)$ is standard $\}$. We may assume that $r_3 < s_2/2$. Then $B_{r_3}(y)$ looks like the following picture :



Let α be a homotopy class of a loop indicated in the picture. α represents some transformation g in the holonomy group. Let ℓ be the axis of g in case g is not parabolic. Suppose that either g is parabolic or $d(p, \ell) \geq s_2/2$. Then develop C centered at y. The faces Q and Q' of P_y containing the developed images of p becomes almost parallel when r_3 is very small. In particular, if r_3 is not bounded by some constant, then the volume of $B_{r_3+R+K_2}(y)$ approaches to zero when r_3 goes to zero. This is in contradiction with the inclusion of a standard ball within a small distance from y. Thus we may assume that $d(p, \ell) < s_2/2$. Then ℓ is freely homotopic to α in C because of the distant assumption. In particular g is not elliptic. Thus by developing C centered at a point on ℓ, we get a thin fundamental domain. The length of ℓ cannot be arbitrarily small by the same reason. We thus get a lower bound of r_3 which depends only on R, K_1 and K_2. This finally completes the proof.

We now follow the notation in the end of § 3. Namely we have a positive increasing sequence t_n which converges to t_0 where a cone-manifold structure degenerates. That means C_n converges to some degenerate metric space in Gromov's sense.

Since we will deal with the case when thin part does not vanish, by rearranging a sequence, we may assume that $C_{n,\mathrm{thin}(1/n)} \neq \phi$. Choose a base point x_n from the $1/n$-thin part of C_n. Rescaling C_n by a homothety of multiplying $1/\mathrm{inj}(x_n)$, we get a cone-manifold $(\bar{C}_n, \bar{x}_n)$ of constant curvature $-(\mathrm{inj}(x_n))^2$. Notice that $\mathrm{inj}(\bar{x}_n) = 1$.

<u>Proposition 6.2</u> : <u>There is a subsequence</u> $\{k\} \subset \{n\}$ <u>so that</u> $(\bar{C}_n, \bar{x}_n)$ <u>converges to a complete euclidean</u> 3-<u>cone</u>-<u>manifold</u> (E, y) <u>other than one listed in Proposition 5.1</u>.

<u>Proof</u>. By the previous proposition, given $R > 0$, there is δ so that $(B_R(\bar{x}_n, \bar{C}_n))_{\mathrm{thin}(\delta)} = \phi$. Think of a sequence of cone-

manifolds $\{B_R(\bar{x}_n, \bar{C}_n)\}$. Then by the same way as in the proof of Theorem 4.2, there is a subsequence which converges to a euclidean 3-cone-manifold with boundary. Since R can be taken arbitrarily large, we have the limit euclidean 3-cone-manifold (E, y) by the diagonal argument. Since $\mathrm{inj}(y)$ must be equal to 1, E cannot be one listed in Proposition 5.1.

§ 7 When thick part live

In this section, we deal with the case when there is ε so that $C_{n,\mathrm{thick}(\varepsilon)} \neq \phi$ for all n.

Choose a base point x_n in the thick part of C_n. Then slightly modifying the argument in the proof of Theorem 4.2, we get a subsequence $\{k\} \subset \{n\}$ so that (C_k, x_k) converges to a complete hyperbolic 3-cone-manifold (C^*, x). Thus we assume from the beginning that (C_n, x_n) converges to (C^*, x) and see how the end of C^* looks like in the below.

Case (i) : each component A_n of Σ_n stays in the thick part or tends to infinity. More precisely, there is $\delta > 0$ and $n_0 > 0$ so that A_n is in the δ-thick part of C_n for all $n > n_0$ or for any $\delta > 0$, there is n_0 so that A_n does not intersect to the δ-thick part of C_n for $n > n_0$. Then choose a decreasing sequence $\{\delta_n\}$ which converges to 0 and let B_n be the difference $C_{n,\mathrm{thin}(\delta)} - C_{n,\mathrm{thin}(\delta_n)}$. Then C_n converges to C^* and the δ-thick part converges to the δ-thick part of C^* and B_n converges to B where B is the δ-thin part of C^*. Since B does not intersect with the singular locus, it consists of finitely many toral cusps. We will see in Proposition 7.2 that this case cannot occur in fact.

Case (ii) : otherwise. That is to say, there are a subsequence $\{j\} \subset \{n\}$ and a decreasing sequence $\{\delta_j\}$ and a component A_j of Σ_j so that $\partial(C_{j,\mathrm{thick}(\delta_j)}) \cap A_j \neq \phi$. Choose a base point y_j on $\partial(N_{j,\mathrm{thick}(\delta_j)}) \cap \Sigma_j$. Then by the similar method in the proof of Proposition 6.2, we have a further subsequence satisfying

<u>Proposition 7.1</u> : <u>There is a subsequence</u> $\{k\} \subset \{n\}$ <u>so that</u> $(\bar{C}_k, \bar{y}_k)$ <u>converges to a noncompact euclidean</u> 3-<u>cone</u>-<u>manifold</u> (E, y) <u>with non empty singular locus</u>. E <u>has the properties that every cone angle of the singular loci is</u> $\leq \pi$ <u>and that</u> $\mathrm{inj}(y) = 1$.

Recall that the fundamental domain of a noncompact euclidean 3-cone-manifold E of cone angles $\leq \pi$ has one of the following forms. Type a) (compact 1-dimensional) × (noncompact 2-dimensional). Type b) (noncompact 1-dimensional) × (compact 2-dimensional). We further split Type a) into two cases.

(a-1) When (E, y) is isometric to (2-dimensional open disk with a cone of angle $\leq \pi$) × S^1. For given $R > 0$ and $\varepsilon > 0$, there is k_0 so that there is an ε-approximation between $(B_R(\bar{y}_k, \bar{C}_k), \bar{y}_k)$ and $(B_R(y, E), y)$ for all $k > k_0$. They actually homeomorphic each other for sufficiently large k by the same argument in the end of § 4. Take R to be very large. Then since $\mathrm{inj}(\bar{y}_k) = 1$, $\mathrm{inj}(y) = 1$ and hence Σ has length 1. Since y is on Σ, $B_1(y, E)$ is standard, that implies $B_1(\bar{y}_k, \bar{C}_k)$ being standard. Hence for any $\bar{z} \in B_1(\bar{y}_k, \bar{C}_k)$, we have $\mathrm{inj}(\bar{z}) \geq 1$. This means that $\mathrm{inj}(z) \geq \delta_k$ for any $z \in B_{\delta_k}(y_k, C_k)$. On the other hand, since y_k was on the boundary of the δ_k-thin part of C_k, there is $z \in B_{\delta_k}(y_k, C_k)$ so that $\mathrm{inj}(z) < \delta_k$. This is contradiction.

(a-2) When there is an edge of Σ_E whose cone angle is π. This means that the deformation along the path reaches to the final target and C^* is a complete hyperbolic 3-orbifold of finite volume. Thus by a theorem of Margulis, there is $\delta > 0$ so that the δ-thin part consists of finitely many cusps.

We now split Type b) into two cases.

(b-1) $S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1$ where $\alpha+\beta+\gamma = 2\pi$. We may assume that $t_0 < 1$ because of the argument in (a-2). We further split this case into two cases.

(b-1-i) When for given $\delta > 0$, there is k_0 so that the δ-thin part of C_k contains noncuspidal component E_k for all $k > k_0$. Choose a base point y_k on ∂E_k. Then $(\bar{C}_k, \bar{y}_k)$ converges to a noncompact euclidean 3-cone-manifold. Assume that $E = S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1$ as in the assumption of the case (b-1). If we take k and R to be sufficiently large, then $B_R(\bar{C}_k, \bar{y}_k)$ is homeomorphic to $B_R(S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1)$ which is homeomorphic to $S^2(\alpha,\beta,\gamma) \times [a,b]$. Since $\alpha+\beta+\gamma = 2\pi$ at $t = t_0 < 1$, the sphere of these three cones originated from a spherical suborbifold in O (notice that the sum of three cone angles at the final destination is supposed to be greater than 2π). Hence by irreducibility of O, the sphere bounds a standard cone ball and in particular three axis meet at the vertex. This means that E_k is a cusp which contradicts the assumption of (b-1-i), and hence this cannot occur.

(b-1-ii) When there is $\delta > 0$ and a subsequence $\{j\} \subset \{k\}$ so that the δ-thin part of C_j consists of cusps for all j. Then the cusp can be deformed furthermore, which means C^* is not a degenerate geometric limit.

(b-2) When there is an edge of Σ_E whose cone angle is equal to π. Then by the same argument of (a-2), this is the final destination.

Summarizing those, we had seen

Proposition 7.2 : The end of C^* consists of finitely many cusps.

Let C' be the closure of C^* - cusps.

Proposition 7.3 : Under the assumption of this section, C' can be embedded in O as a suborbifold whose boundary is incompressible.

Proof. We have a homeomorphism $f_k : (X_k, \Sigma_k) \to (X_0, \Sigma_0)$ whose homotopy class is canonically determined by the original continuous algebraic deformation of the holonomy. Choose δ so that C' is the δ-thick part of C^*. Then there is an approximation between C' and the δ-thick part of C_k. Choose this by an into homeomorphism $\phi_k : (X_{C'}, \Sigma_{C'}) \to (X_k, \Sigma_k)$ for sufficiently large k and moreover such that the images of $f_k \cdot \phi_k$ stay constant in X_0 while k varies. (We assumed here that this can be done since it sounds reasonable, however we may have to prove it.) Passing subsequences, we may further assume that $f_k \cdot \phi_k$ maps a component of $\partial X_{C'}$ to a constant surface in X_0 while k varies. Identfying $X_{C'}$ with the image of $f_k \cdot \phi_k$'s, we regard $X_{C'}$ as a subspace of X_0. Then the map $f_k \cdot \phi_k$ will be a self homeomorphism of $X_{C'}$ leaving the component of boundary invariant. We should notice here that the homotopy class of $f_k \cdot \phi_k$ depends on what k is. From now on, k is always assumed to be large enough.

We have already shown that a component of $\partial C'$ is either a torus, $S^2(\pi,\pi,\pi,\pi)$ or $S^2(\alpha,\beta,\gamma)$ with $\alpha+\beta+\gamma = 2\pi$. We will prove this proposition by showing that $X_{C'}$ does not have a toral boundary, C' is embedded in O by $f_k \cdot \phi_k$ as a suborbifold and it has an incompressible boundary.

Let us first show that C' does not have a toral boundary. Suppose $X_{C'}$ has a toral boundary which is not ∂-parallel but incompressible in $X_O - \Sigma_O$, then it contradicts to the simplicity of our initial setting. Hence we can assume that all of toral boundary is either ∂-parallel or compressible in $X_O - \Sigma_O$. Choose a a loop ℓ on a toral boundary of $X_{C'}$ so that it bounds a disk possibly with a cone in X_O. Since it bounds a disk in X_O, the holonomy ρ_k of C_k must map $f_k^{-1}(\ell)$ to a trivial element or an elliptic element with bounded rotation angle. We denote $(f_k \cdot \phi_k)^{-1}(\ell)$ by ℓ_k. Notice that every ℓ_k represents a parabolic element in C'. Suppose that there is a subsequence $\{i\} \subset \{k\}$ so that all ℓ_i represent the unique class ℓ^* in $\pi_1(X_{C'} - \Sigma_{C'})$. This means that $f_i^{-1}(\ell)$ converges to the loop representing this class. In particular, $\rho_i(f_i^{-1}(\ell))$ must converge to $\rho(\ell^*)$ where ρ is a holonomy of C'. However this is contradiction since $\rho_i(f_i^{-1}(\ell))$'s are trivial or elliptic of bounded rotation angle and cannot converge to a parabolic element $\rho(\ell^*)$. Hence we have the other case. That is, ℓ_k all are distinct in $\pi_1(X_{C'} - \Sigma_{C'})$. Then think of a homeomorphism $g_j : X_{C'} \to X_{C'}$ defined by $(f_j \cdot \phi_j)^{-1} \cdot f_k \cdot \phi_k$ for $j > k$. Obviously $g_j(\ell_k) = \ell_j$. Take a double of $X_{C'}$ along its sphericl boundary and denote it by $DX_{C'}$. Then $M = DX_{C'} - D\Sigma_{C'}$ is an irreducible and Haken manifold and hence it admits a geometric decomposition. By doubling the maps g_j's, we get a family of self homeomorphisms of M which leaves the component of the boundary invariant. Futhermore the restriction of the family to the components of ∂M containing a copy of ℓ provides infinitely many distinct homotopy classes as maps. This can happen only when a geometric piece of M containing a copy of ℓ on its boundary is a Seifert fibered space S. Moreover S must have another boundary component which is a boundary of M and to where the restriction of our homeomorphisms provides infinitely many homotopy classes. This is because the restriction of a homeomrphism of M to the torus appeared in the torus decomposition can provide only finitely many homotopy classes. Now, the other boundary component either inherits a toral boundary of $X_{C'}$ or a boundary of a tubular neighborhood of a component of $D\Sigma_{C'}$. In the second case, a fiber of a Seifert fibration restricted to $\partial\mathcal{N}(D\Sigma_{C'})$ must be homotopic to a meridional loop since otherwise we cannot have infinitely many homotopy classes of maps which extend to a map of $DX_{C'}$. We now have a saturated essential annulus A connecting these two boundaries. Cutting A along the central surface of M which was the spherical boundary of $X_{C'}$, we get as a part of A an essential annulus A' in $X_{C'} - \Sigma_{C'}$. One component of $\partial A'$ is on the toral boundary of $X_{C'} - \Sigma_{C'}$ containing ℓ. This represents a parabolic element in the holonomy of C'. The other component is either on a toral boundary of $X_{C'} - \Sigma_{C'}$ or on a spherical boundary of $X_{C'}$. Thus in any case, it represents a parabolic or an elliptic element in the holonomy of C'. Hence A' turns out to be an essential annulus in $X_{C'} - \Sigma_{C'}$ which joins a cusp with either a distinct cusp or an elliptic axis in C'. This is imposssible and hence C' cannot have a toral boundary.

Now, $\partial C'$ consists of $S^2(\pi,\pi,\pi,\pi)$'s and $S^2(\alpha,\beta,\gamma)$'s where $\alpha+\beta+\gamma = 2\pi$. We next show that the inclusion $X_{C'} \subset X_O$ becomes an inclusion of orbifolds $C' \subset O$. To see this, assume first that $\partial C'$ consists only of $S^2(\alpha,\beta,\gamma)$'s where $\alpha+\beta+\gamma = 2\pi$ and none of α', β, γ's is equal to π. Then every component of $\partial X_{C'}$ must bound a cone ball in X_O since O is irreducible. By Theorem 2.4 and 2.6, we can further deform this cone-structure towards the final target. This means that C' is not the geometric limit, which is contradiction. Thus we have at least one cone point on $\partial C'$ with angle $= \pi$. This shows that our deformation reaches to the final destination. Hence a cone-manifold C' is an orbifold and the inclusion $X_{C'} \subset X_O$ by the map $f_k \cdot \phi_k$ supports an embedding of C' to O as orbifolds.

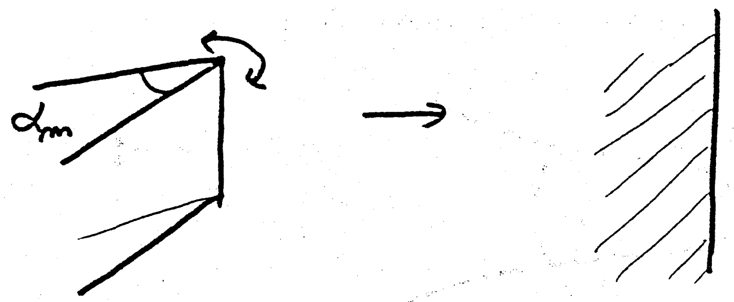
Let us identify C' with the image of the embedding to O and regard C' as a suborbifold of O. Last of all, we show that $\partial C'$ is incompressible. The argument is quite similar to the above one. Assume if not, then there is a disk possibly with a cone in X_O whose boundary is a nontrivial and nonperipheral loop ℓ on $\partial X_{C'} - \Sigma_{C'}$. Thus this represents a parabolic element in C'. Since it bounds a disk possibly with a cone in X_O, the holonomy ρ_k of C_k must map $f_k^{-1}(\ell)$ to either a trivial element or an elliptic ellement of bounded rotation angle. We again denote $(f_k \cdot \phi_k)^{-1}(\ell)$ by ℓ_k. Suppose that there is a subsequence $\{i\} \subset \{k\}$ so that all ℓ_i's represent the unique class ℓ^* in $\pi_1(X_{C'} - \Sigma_{C'})$. Then, if we let ρ be the holonomy C', the sequence $\rho_i(f_i^{-1}(\ell))$ must converge to $\rho(\ell^*)$ which is a parabolic element. However this is impossible. Hence we have the other case. That is, all ℓ_k's represent distinct classes in $\pi_1(X_{C'} - \Sigma_{C'})$. Define a homeomorphism $g_j : X_{C'} \to X_{C'}$ by $(f_j \cdot \phi_j)^{-1} \cdot f_k \cdot \phi_k$ for $j > k$. Notice that $g_j(\ell_k) = \ell_j$. By simple argument, we can see that $M = DX_{C'} - D\Sigma_{C'}$ is irreducible and atoroidal. Thus it admits a complete hyperbolic structure of finite volume by the uniformization theorem. On the other hand, we can construct infinitely many homotopy classes of self homeoemorphisms of M by doubling g_j. This is again contradiction and we have shown that $\partial C'$ cannot be compressible.

We have shown up to here that we get a hyperbolic 3-suborbifold C' of O and a torus decomposition of O. Since Haken's finiteness theorem even holds for this situation, we can reduce the argument to the other case by induction of the maximal number of components splitted by incompressible euclidean suborbifolds.

§ 8 When thick part die

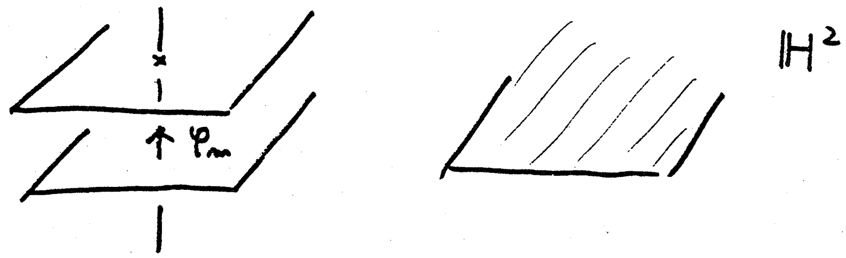
We first see how geometric limits without rescaling look like. We see it locally by thinking of the geometric limits of $H^3/\langle\phi_m\rangle$ where $\{\phi_m\}$ is a sequence of elements of Isom^+H^3 which converges to the identity element.

Case 1. When ϕ_m's are elliptic, then its limit is a half-plane.

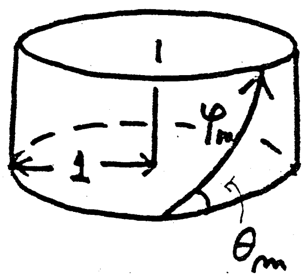


<u>Case 2</u>. When ϕ_m's are parabolic, then its limit is $\mathbb{H}^2$.

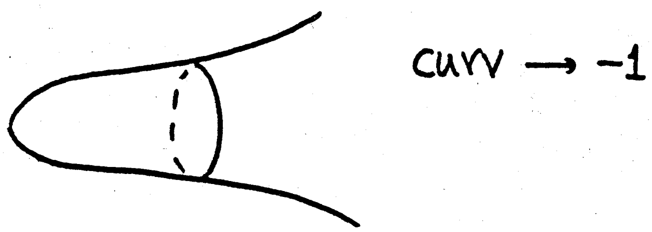
<u>Case 3</u>. When ϕ_m's are hyperbolic, then again its limit is $\mathbb{H}^2$.



<u>Case 4</u>. When ϕ_m's are loxodromic, then there are mainly three cases. To see this, let us define the twisting angle θ_m of ϕ_m by



If θ_m approaches $\pi/2$, then its limit is $\mathbb{H}^2$, if θ_m approaches 0, then its limit becomes a half-line. Otherwise, the limit is homeomorphic to $\mathbb{R}^2$ whose metric looks like



The computation of the metric on the limit shows that it is a surface of revolution.

Go back to our situation, that is, for any $\varepsilon > 0$, there is n_0 so that the ε-thick part of C_n is empty for all $n \geqq n_0$. Then we may choose a decreasing sequence $\{\varepsilon_n\}$ converging 0 so that the ε_n-thick part of C_n is empty. Choose a base point $x_n \in C_n$. Since the ε_n-thick part of C_n is empty, if $\varepsilon > \varepsilon_n$, then $B_\varepsilon(x_n, C_n)$ is not standard. We then have two cases a) $B_\varepsilon(x_n, C_n) \cap \Sigma_n \neq \phi$ and b) $B_\varepsilon(x_n, C_n) \cap \Sigma_n = \phi$.

We will see here that if b) is the case, then $(\bar{C}_n, \bar{x}_n)$ converges to a noncompact euclidean 3-manifold (E, y) with $\mathrm{inj}(y) = 1$. The existence of the limit has been already done. E does not have singularity since $B_\varepsilon(x_n)$ and hence $B_{\varepsilon/\mathrm{inj}(x_n)}(\bar{x}_n, \bar{C}_n)$ does not have singularity. E cannot be compact since otherwise, $(X_{\bar{C}_n}, \Sigma_{\bar{C}_n})$ is homeomorphic to (X_E, Σ_E) for sufficiently large n and hence E must contain the singularity. This is contradiction.

We thus have a fairly clear picture for E. That is either $\mathbb{E}^3/\mathbb{Z}$, $\mathbb{E}^3/\mathbb{Z}+\mathbb{Z}$ or $\mathbb{E}^3/\pi_1$(Klein bottle).

<u>Proposition 8.1</u> : <u>For any</u> $R > 0$ <u>and</u> $\varepsilon > 0$, <u>there is</u> δ <u>which depends only on</u> ε, R (<u>and does not depend on</u> n) <u>so that</u> (1) $R\delta < \varepsilon$ <u>and</u> (2) <u>if</u> $x \in C_{n,\mathrm{thin}(\delta)}$ <u>and</u> $B_\varepsilon(x, C_n)$ <u>is a proper subset of</u> C, <u>then</u> $(B_{R\cdot \mathrm{inj}(x)}(x, C_n), x)$ <u>is homeomorphic to</u> $(B_{R\cdot\mathrm{inj}(y)}(E), y)$ <u>for some noncompact euclidean</u> 3-<u>cone-manifold</u>.

<u>Proof</u>. We are given R and ε. Let us assume contrary that for any $\delta = 1/k$ satisfying $R/k < \varepsilon$, there is $x_k \in C_{n_k}$ with

$\mathrm{inj}(x_k) < 1/k$ so that an $R\cdot\mathrm{inj}(x_k)$-neighborhood of x_k in C_{n_k} never be homeomorphic to $B_{R\cdot\mathrm{inj}(y)}(y, E)$ for a noncompact euclidean 3-cone-manifold E and a base point y. Let $(\bar{C}_{n_k}, \bar{x}_k)$ be a metric space obtained by rescaling C_{n_k} by multiplication of $1/\mathrm{inj}(x_k)$. Then $\mathrm{inj}(\bar{x}_k) = 1$ and sectional curvature is $= -\mathrm{inj}(x_k)^2$. Then the sequence $(B_{\varepsilon/\mathrm{inj}(x_k)}(\bar{x}_k, \bar{C}_{n_k}), \bar{x}_k)$ converges to some noncompact euclidean 3-cone-manifold (E, y) with $\mathrm{inj}(y) = 1$. If k is large enough, then $\varepsilon/\mathrm{inj}(x_k) > \varepsilon k > R$, and hence $(B_R(\bar{x}_k, \bar{C}_{n_k}), \bar{x}_k)$ is homeomorphic to $B_R(E, y)$ for further large k. The first one is homeomorphic to $(B_{R\cdot\mathrm{inj}(x_k)}(x_k, C_{n_k}), x_k)$, and we get contradiction.

<u>Corollary 8.2</u> : <u>For any</u> $R > 0$ <u>and</u> $\varepsilon > 0$, <u>there is</u> $\delta > 0$ <u>so that if</u> $x \in C_{n,\mathrm{thin}(\delta)}$ <u>and</u> $B_\varepsilon(x) \cap \Sigma_n = \phi$, <u>then</u> $\pi_1(B_{R\cdot\mathrm{inj}(x)}(x, C_n))$ <u>is isomorphic to either</u> $\mathbb{Z}$, <u>or</u> $\mathbb{Z}+\mathbb{Z}$.

<u>Proof</u>. This is a direct corollary to Proposition 8.1 except for the fact that the Klein bottle group does not appear. Assume that π_1 is isomorphic to the Klein bottle group. Then $C_n \supset C_{n,\mathrm{thin}(\delta)}$ containes a twisted I-bundle over the Klein bottle K. Since $X_n - \Sigma_n$ was simple, ∂K must either be parallel to a component of $\partial\mathcal{N}(\Sigma_n)$ or bound a solid torus in $X_n - \Sigma_n$. In both cases, we get contradiction with the initial setting.

Let us call $B_{R\cdot\mathrm{inj}(x)}(x, C_n)$ a local neighborhood of x and its fundamental group a local π_1 of x. Fix $R > 0$ very large and take a decreasing sequence $\{\varepsilon_n\}$. Then for any $m > 0$, there is $n_m > 0$ so that $\delta(1/m, R) > \varepsilon_{n_m}$. We remark here that δ is less than $1/Rm$ and hence $1/m$. To make notation simple, let us denote the new sequence $\{C_{n_m}\}$ by $\{C_m\}$. In relating notions, we replace all of n_m by m.

Let us denote $C_m - \mathcal{N}_{1/m}(\Sigma_m)$ by C'_m. Since $1/m > \delta > \varepsilon_m >$ inj(x) for any $x \in C_m$, the neighborhood of Σ_m we take is fairly thick. We then have two cases.

Case a) $C'_m \neq \phi$ for infinitely many m. Then the geometric limit of (C'_m, x_m), denoted by (C^*, x), is isometric to the geometric limit of the original sequence (C_m, x_m) since there is a 1/m-approximation between C_m and C'_m for each m.

Case b) There exists m_0 so that $C'_m = \phi$ for all $m > m_0$. Then $\mathcal{N}_{1/m}(\Sigma_m)$ and hence Σ_m converges to the geometric limit C^* which is expected to have dimension at most one.

To see what happens more precisely, we analyze a relation between O and C^* by assigning some foliation structures on C'_m and $\mathcal{N}_{1/m}(\Sigma_m)$.

For each point of C'_m, we associated the local fundamental group which is generated by nearly straight short loops. Thus each generator corresponds to a nontrivial element in the holonomy group. Let us foliate this region C'_m first.

Think first of the part whose local fundamental group is $\mathbb{Z}+\mathbb{Z}$. Since abelian group of rank two in $PSL_2\mathbb{C}$ must be generated by two parabolic elements with the same fixed point at ∞, or two loxodromic elements with common axis. Thus in both cases, we have a canonical neighborhood consisting of a family of equi-distant euclidean tori from the fixed point or the axis respectively. Foliate this part by these tori.

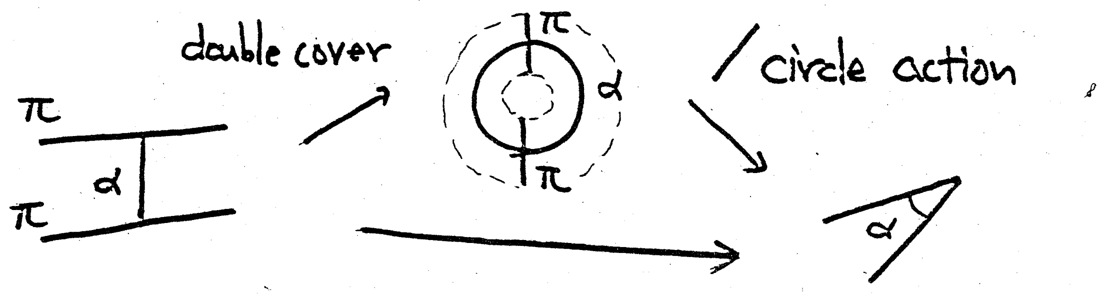
The remaining is the part of which the local π_1 is $\mathbb{Z}$. First, we show that a component of this part has a unique generator for the local fundamental groups. This means that any local π_1 is generated by some power of the unique element in the holonomy group. Because, for any x in this part, a loop ℓ_x representing a gnerator of a local fundamental group in a local neighborhood represents a nontrivial element in the holonomy group. If we let α be its primitive element in the holonomy group, α has a family of equi-distant euclidean tori from its cusp or axis according to whether it is parabolic or not. We may assume that ℓ_x is homotopic to a loop on this torus. If we take x' in a small neighborhood of x, then the corresponding loop $\ell_{x'}$ is also homotopic to a loop on the parallel torus to one for ℓ_x. Since $\ell_{x'}$ is nontrivial in the holonomy group, it represents some power of α. By taking the method analogous to analytic continuation, we can see that α is a unique element for generating each local fundamental group within a component of the part of local $\pi_1 = \mathbb{Z}$.

For each nontrivial element in the holonomy, we have an equi-distant euclidean torus in C_m. For a component of the part whose local π_1 is $\mathbb{Z}$, we have associated the unique nontrivial element α in the holonomy group. Let us first foliate this part by equi-distant tori of α. Then choose one torus whose diameter is greater than ε_m. Such one must exist since the local fundamental group is generated by one element. Foliate this torus by intrinsic shortest geodesics. Then extend this foliation to all of the part. The extension may have singular leaf as in the Seifert fibration at the axis of α.

We have to worry about another singular fashon. That is, a equi-diatant torus may not be embedded in C'_m. We will see that this cannot occur in fact. The equi-distant tori form a continuous family which starts from the cusp or the axis of α according to whether α is parabolic or not. In both cases, we have an embedded torus near the start position. Hence if it will be a singular torus as growing, there must be a critical equi-distant torus which touches to itself in C'_m. Take a base point near the start position and think of a based loop which stays in a family of equi-distant tori up to the critical time and passes through the touched point. Let β be its homotopy class. Now let ℓ be a loop which represents the local π_1 of the touched point. Since ℓ was on this critical torus, ℓ and β must commute in the holonomy group. In particular, α and β commute and hence they have an common axis. This shows that the action by β cannot produce a touched point of an equi-distant torus of α. Thus the 1-dimensional foliation we constructed becomes well-defined.

Summarizing the above, we get a foliation on C'_m by tori, circles and a part of them.

We next construct a 1- and 2-dimensional foliation on $\mathcal{N}_{1/m}(\Sigma_m)$. Since it is a part of C_m, the injectivity radius is bounded by ε_m for any point in there. By Proposition 8.1, each point there has a small but not very small neighborhood which is homeomorphic to a neighborhood of some noncompact euclidean 3-cone-manifold E. Also this homeomorphism can be chosen almost conformal. That is to say, each cone angles of Σ_m is very near to that of Σ_E. Since our not small neighborhood is not standard, E has a fibered structure. Furthermore if there is an edge of Σ_E whose cone angle is π, we may assume that E is an euclidean orbifold since that is the final destination of our deformation. We can check that this is the case by the classification of noncompact euclidean cone-manifolds. Hence a fibration becomes a fibration of orbifolds. The following is the local picture of fibrations to a 2-dimensional object :



Choose a circle fibration in this case so that a fiber is the shortest loop on the torus illustrated in the above picture. The followings are local pictures of fibrations to the 1-dimensional objects :

$$S^2(\pi,\pi,\pi,\pi)\times\mathbb{E}^1 \qquad\qquad S^2(\alpha,\beta,\gamma)\times\mathbb{E}^1$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad \alpha+\beta+\gamma=2\pi$$
$$\mathbb{E}^1 \qquad\qquad\qquad\qquad\qquad \mathbb{E}^1$$

Thus pulling back these local foliations to $\mathcal{N}_{1/m}(\Sigma_m)$ by approximation homeomorphisms, we get a foliation there. We actually need some compatible arrangement of the foliation along paching part, however this looks easy.

We thus got foliations on C'_m and $\mathcal{N}_{1/m}(\Sigma_m)$. Let D_m and J_m be the orbit spaces of such foliations on C'_m and $\mathcal{N}_{1/m}(\Sigma_m)$ respectively. Both D_m and J_m admit a metric naturally induced by the leaf distance. The set of $x \in D_m$ contained in the δ-standard ball is called the δ-thick part of D_m. We can easily verify that there is an ε_m-approximation between D_m and C'_m. There is also a $1/m$-approximation between C_m and C'_m. Hence we have an (ε_m+1/m)-approximation between D_m and C_m and therefore D_m converges to the geometric limit C^* of C_m. The geometric limit of J_m is obviously a part of C^* provided the distance between $C_{m,thick}$ and Σ_m is uniformly bounded, because of the uniqueness of the limit. When C^* is of 2-dimesional and with boundary, the boundary is a part of the geometric limit of J_m.

Now, what follows is the unfortunate part of this note. Since there are several assertions of which we have not been able to understand, we just give up to check details and describe only an ontline. However, since we write several paragraphs without being convinced, it may be unreasonable even as an outline. Hence we wish the reader to regard it only as our working hypothesis.

<u>Proposition 8.3</u> : <u>The</u> 2-<u>dimensional</u> <u>area</u> <u>of</u> D_m <u>is</u> <u>bounded</u> <u>by</u> <u>a</u> <u>constant</u> B <u>for</u> <u>all</u> m.

<u>Proof</u>. ? ? ?

Assumeing this proposition, we go forward.

Case a) When there are $\varepsilon > 0$ and m_0 so that the ε-thick part of D_m is nonempty for all $m > m_0$. A component of the thick part corresponds to a component of the part whose local $\pi_1 = \mathbb{Z}$. Since the geometric limit of D_m is the geometric limit C^* of C_m, it is going to be a hyperbolic 2-cone-manifold or a surface of revolution as was observed previously. Notice in this case that if $\partial C^* \neq \phi$, it is a geometric limit of a corresponding component of J_m. Suppose C^* is a surface of revolution, then by its creation manner, the boundary if any must be geodesic and must go out from the center of C^* (recall that ∂ inherits ∂J_m !).

Since there is only one center on C^*, the area of C^* becomes infinity. This contradicts Proposition 8.3. Thus C^* must be a hyperbolic 2-cone-manifold of finite area. Then as we have observed before, the twisting angle of a generator of a local π_1 of C'_m approaches $\pi/2$ and the direction of a fiber in C'_m approaches the direction of a fiber in $\mathcal{N}_{1/m}(\Sigma_m)$ when m tends to infinity. Thus we may assume that we have a fibration $p : X_{C_m} \to C^*$ for sufficiently large m.

Subcase a-1) When the δ-thin part of D_m is empty for all $m > m_0$. Then C^* is a compact hyperbolic 2-cone-manifold with nonempty boundary. Because if it is closed, then a fibration p becomes a Seifert fibration of $N - \Sigma_N$ which contradicts the initial setting. Now since C^* has the boundary, there must be a singular locus whose cone angle is π. Then X_{C_m} is homeomorphic to X_O and p becomes a Seifert fibration of orbifolds : $O \to C^*$.

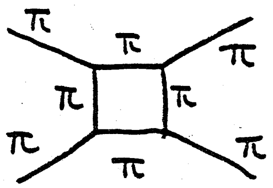
Subcase a-2) When for any $\delta > 0$, there is m so that the δ-thin part of D_m is nonempty. Then C^* is a complete hyperbolic 2-cone-manifold with cusp. Since the preimage of cusp has a circle fibration, it must be either a torus cusp or its quotient by fiber preserving involution. The section is $S^2(\pi,\pi,\pi,\pi)$. Suppose either one of the ends is the last one or C^* has a boundary, then by the same reason as above, p supplies a Seifert fibration of the orbifold which is the final destination. So we assume that C^* has no boundary and every cusp corresponds to a toral end. Then actually $N - \Sigma_N$ admits a Seifert fibration which contradicts the initial setting.

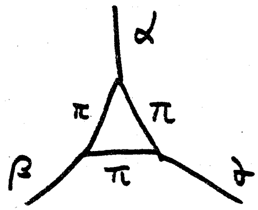
Thus our fibered structure on $X' = X_{C_m} - p^{-1}(\text{cusps})$ becomes an orbifold fibration of the final destination. Hence X' can be regarded as an underlying space of a proper suborbifold O' of O at the beginning. Suppose $\partial O'$ is incompressible in O, then it gives a torus for the torus decomposition. If it is compressible, the fibration on $\partial O'$ extends to a fibration on Dehn filling resultant orbifold. Thus again we get a Seifert fibration on $O' \cup$ (solid torus) $\subset O$.

Case b) When for any $\varepsilon > 0$, there is m_0 so that the ε-thick part of D_m is empty for all $m > m_0$. This is equivalent to say that C'_m is thin for all $m > m_0$. Thus C^* will be of at most 1-dimensional. Since the sequence of rescaling cone-manifolds $\bar{C}_m$ converges to a noncompact euclidean cone-manifold with at most two ends, C^* will be a manifold. This means that C^* is either a circle, a closed interval, a half open interval, an open interval or a point. By Proposition 8.1, a neighbrhood of any point in X_m is homeomorphic to $S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1$ with $\alpha+\beta+\gamma = 2\pi$ or $S^2(\pi,\pi,\pi,\pi) \times \mathbb{E}^1$ for sfficiently large m. Thus we again have a fibration $p : X_{C_m} \to C^*$ for sufficiently large m.

Subcase b-1) When C^* is a circle. A neighborhood of p^{-1}(point) was homeomorphic to $S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1$ with $\alpha+\beta+\gamma = 2\pi$ or $S^2(\pi,\pi,\pi,\pi) \times \mathbb{E}^1$. The first case cannot happen since every homeomorphisms of $S^2(\alpha,\beta,\delta)$ is homotopic to a periodic map and hence $N - \Sigma_N$ admits a Seifert fibration which contradicts the initial setting. Thus we do have only the last case. This shows that we have reached the final destination since it contains a singular locus of cone angle π. This can be seen to be a solvable orbifold because if not, O must be an euclidean or nilpotent orbifold and hence $N - \Sigma_N$ contains an essential torus, which contradicts the initial setting.

Subcase b-2) When C^* is not a circle but 1-dimensional. If C^* is a closed interval, then the fibration gives an euclidean 2-orbifold bundle over I. If C^* is a half open interval, then N should be the followings :





∴ euclidean of vol = ∞

If C^* is an open interval, then we have

$$S^2(\pi,\pi,\pi,\pi) \times \mathbb{E}^1$$

$$S^2(\alpha,\beta,\gamma) \times \mathbb{E}^1$$

∴ not occur

Subcase b-3) When C^* is a point. Then again enlarging J_m to $\bar{J}_m$ so that diam $\bar{J}_m = 1$.

b-3-1) When there is $\varepsilon > 0$ so that $\bar{J}_{m,\mathrm{thick}(\varepsilon)} \neq \phi$ for all m. Then by Proposition 6.1, there is $\delta > 0$ so that $\bar{J}_{m,\mathrm{thin}(\delta)} = \phi$. Thus $\bar{J}_m$ converges to a compact euclidean 3-cone-manifold E. If $t_0 = 1$, then E is nothing but O. Even if $t_0 < 1$, the combinatorial type of E is the same as that of O.

<u>Theorem 8.4</u> : <u>If $t_0 < 1$, there is a deformation of a euclidean metric on E to a metric of O with positive Ricci curvature</u>.

<u>Proof</u>. ? ? ?

Assume this theorem, then by the orbifold version of Hamilton's theorem [H], such a metric on O can be deformed to a metric of constant positive curvature. That is to say, O is a spherical orbifold.

b-3-2) When for any ε, there is m_0 so that $\overline{J}_{m,\text{thick}(\varepsilon)} = \phi$ for all $m > m_0$. Then $\overline{J}_m$ converges to an euclidean cone-manifold E as in the case a).

When E is an euclidean 2-cone-manifold, then $t_0 = 1$ and O must be an euclidean or a nilpotent orbifold. Notice that E cannot be a surface of revolution since $\operatorname{diam} E = 1$. When E is one dimensional, then E in this case must be a closed interval or a circle. In any case, O is a solvable orbifold.

References

[CS] Culler, M. and Shalen, P.: Varieties of group representations and splittings of 3-manifolds, Ann. of Math., 117 (1983), 109 - 146.

[G] Gromov, M.: Structures métriques pour les variétés riemanniennes, Cedic-Fernand-Nathan, 1981.

[H] Hamilton, R.: Three-manifolds with positive Ricci curvature, J. Differential Geometry, 17 (1982), 255 - 306.

[K] Kojima, S.: A construction of geometric structures on Seifert fibered spaces, J. Math. Soc. Japan, 36 (1984), 483 - 495.

[L] Lok, L.: Thesis, Columbia University, 1984.

[MG] Montgomery, D. and Zippin, L.: Topological transformation groups, Interscience Tracts, No 1, 1955.

[S] Scott, P.: The geometries of 3-manifolds, Bull. London Math. Sc., 15 (1983), 401 - 487.

[Sm] The Smith conjecture, edited by J. Morgan and H. Bass, Acad. Press, 1984.

[T] Thurston, W.: The geometry and topology of 3-manifolds, Lecture note, Princeton University 1978.

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